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A RENEWAL THEORY WITH VARYING DRIFT

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Let \( R \) be the excess over the boundary in renewal theory. It is well known that \( ER \) has a limit \( r \) when the drift of the random walk \( \mu \geq 0 \). We study renewal theorems with varying \( \mu \). Conditions are given under which the tail \( ER - r \) is uniformly dominated by a decreasing integrable function for \( \mu \) in a compact interval in \((0, \infty)\). Conditions are also given under which the derivative of the tail \( \partial / \partial \mu |_{ER - r} \) is uniformly dominated by a directly Riemann integrable function.

1. Introduction. Let \( X, X_1, X_2, \ldots \) be i.i.d. random variables with \( E X = 0 \) and \( \text{Var}(X) = \sigma^2 > 0 \). Put \( S_n = X_1 + \cdots + X_n, \ n \geq 1 \). Define for each \( c \geq 0 \) and \( \mu \geq 0 \),

\[
T = T(c, \mu) = \inf\{n: S_n + n\mu > c\}
\]

and

\[
R = R(c, \mu) = S_T + \mu T - c.
\]

For fixed \( \mu \geq 0 \), the asymptotic behavior of \( R \) and \( T \) is well described by the classical renewal theory. See, for example, Feller (1971). This article concerns uniform renewal theorems with varying drift \( \mu \).

Let us first consider the following nonlinear boundary crossing time that motivates our investigation. Define

\[
T_A = \inf\{n: S_n > A(n)\},
\]

where \( A(t) \) is a decreasing function. To approximate probabilistic quantities involving the stopping time \( T_A \) and the overshoot \( S_{T_A} - A(T_A) \) as the boundary tends to infinity, Woodroofe (1976) and Lai and Siegmund (1977, 1979) suggested that we consider the corresponding linear problems involving the stopping time \( T(c_A, \mu_A) \) and the overshoot \( R(c_A, \mu_A) \), where \( c_A = -bA'(b) \), \( \mu_A = -A'(b) \) and \( b = \sup\{t: A(t) = 0\} = ET + O(1) \). Applying classical renewal theorems, they obtained nonlinear renewal theorems for the case \( A'(b) = \text{constant} \). Noting that

\[
T(c_A, \mu_A) = \inf\{n: S_n > A'(b)(n - b)\}
\]

and \( A'(b)(t - b) \) is the Taylor expansion of \( A(t) \) near \( ET \), we realize that a linear renewal theory with varying drift is needed to study more general and deeper nonlinear renewal theorems for two reasons: (a) to study the case

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$A'(b) \neq \text{constant in the limit process}; (b) \text{to get high-order approximations by considering}

\[ T_A^* = \inf\{n: S_n > A^*(n)\}, \]

where $A^*(t)$ is the linear interpolation of $A(t)$. In Zhang (1988) various nonlinear renewal theorems were obtained by using the results in this article.

We shall state the main results in this section and prove Theorems 2 and 3 in Sections 2 and 3, respectively.

A random variable $Z$ is arithmetic if $P\{Z/d = \text{integer}\} = 1$ for some $0 < d < \infty$. In this case, we shall also say that $Z$ has an arithmetic distribution. The span of an arithmetic random variable $Z$, denoted by $\text{span}(Z)$, is the largest real number $d < \infty$ such that $P\{Z/d = \text{integer}\} = 1$. If $Z$ is not arithmetic we define $\text{span}(Z) = 0$. Also, a random variable $Z$ has a lattice distribution if $Z + \mu$ is arithmetic for some real number $\mu$.

Let $Y(\mu), Y_n(\mu), n \geq 1$, be i.i.d. random variables for each $\mu$ with $Y = Y(\mu) = R(0, \mu)$. Define

\begin{align*}
  m_1 &= m_1(\mu) = EY(\mu), \\
  r &= r(\mu) = EY^2(\mu)/(2m_1(\mu)), \\
  G &= G(x, \mu) = \int_x^\infty P\{Y(\mu) > y\} \, dy/m_1
\end{align*}

and

\begin{equation}
  f = f(u, \mu) = E\exp\{iuY(\mu)\}.
\end{equation}

Our first theorem is a uniform Blackwell-type renewal theorem.

**Theorem 1.** Let $\mu > 0$. Suppose that $X + \mu$ does not have an arithmetic distribution. Let $c_k$ and $u_k$ be two sequences of constants such that $c_k \to \infty$ and $0 \leq u_k \to \mu$. Then for any real number $x \geq 0$,

\begin{equation}
  \lim_k \left[ \sum_{n=1}^\infty P\{c_k < Y_1(u_k) + \cdots + Y_n(u_k) \leq c_k + x\} - \frac{x}{m_1(u_k)} \right] = 0,
\end{equation}

\begin{equation}
  \lim_k \sum_{n=1}^\infty P\{c_k < S_n + u_k n \leq c_k + x\} = x/\mu
\end{equation}

and

\begin{equation}
  \lim_k P\{R(c_k, u_k) > x\} = G(x, \mu).
\end{equation}

Furthermore, if $EX^2 = \sigma^2 < \infty$, then for any $x \geq 0$ and $t$,

\begin{equation}
  \lim_k P\{R(c_k, u_k) > x, T(c_k, u_k) > c_k/u_k + t\sqrt{c_k \sigma^2/u_k^3}\}
  = G(x, \mu) P\{N(0, 1) > t\},
\end{equation}

where $N(0, 1)$ is standard normal.
We shall not provide a proof of Theorem 1 here since it can be proved by modifying the existing proofs of the Blackwell renewal theorem. For example, noting that \( \text{span}(X + u_k) \to 0 \), we can modify the methods of Woodroofe (1982), pages 113–116, to prove (1.8) (and therefore the rest of the theorem), since we only need to consider functions \( \hat{g} \) with compact support there. For the nonlattice case Theorem 1 can be easily proved by the coupling methods. See Lindvall (1977). If \( m_1(0) < \infty \), then (1.7) and (1.9) still hold when \( u_k \to 0 \). Sufficient and necessary conditions for \( m_1(0) < \infty \) were obtained by Chow (1986).

Assume that \( \sigma^2 < \infty \). Let \( 0 < a \leq b < \infty \) and

\[
M = \sup \left\{ \sum_{n=1}^{\infty} P(c < S_n + n\mu \leq c + 1) : a \leq \mu \leq b, -\infty < c < \infty \right\}.
\]

By (1.8) \( M < \infty \). Since the right-hand side of

\[
P \{R(c,\mu) > x\} = \sum_{n=1}^{\infty} P \{T(c,\mu) \geq n, S_n + n\mu > x + c\}
\]

\[
\leq \int_{x-b}^{\infty} \sum_{n=1}^{\infty} P \{x + c - y - \mu < S_{n-1} + (n-1)\mu \leq c\} \, dP(X \leq y)
\]

\[
\leq 2M \int_{x-b}^{\infty} (1 + b + y - x) \, dP(X \leq y)
\]

is integrable and independent of \( \mu \), \( \{R(c,\mu) : c \geq 0, a \leq \mu \leq b\} \) is uniformly integrable, so that (1.9) implies \( \lim_{k} ER(c_k, u_k) = r(\mu) \). However, under higher moment and smoothness conditions we have the following stronger results.

**Theorem 2.** Suppose that \( E|X|^3 < \infty \) and for some \( k < \infty \) and \( 0 < a \leq b < \infty \),

\[
(1.11) \quad \sup \left\{ \int_{-\infty}^{\infty} |f(u,\mu)|^k \, du : a \leq \mu \leq b \right\} < \infty.
\]

Then

\[
(1.12) \quad \int_{0}^{\infty} \sup \{|ER(y,\mu) - r(\mu)| : y > x, a \leq \mu \leq b\} \, dx < \infty.
\]

The Lebesgue integrability of \( ER(x,\mu) - r(\mu) \) on \([0,\infty)\) for fixed \( \mu > 0 \) was established by Stone (1965) under the moment condition \( E|X|^3 < \infty \). The smoothness condition (1.11) is similar to the condition C under which a renewal theorem for curved boundaries was obtained by Woodroofe (1976).

A function \( h(\cdot) \) is directly Riemann-integrable if it is Riemann-integrable on any compact interval and

\[
\sum_{n=-\infty}^{\infty} \sup \{|h(x)| : n \leq x < n + 1\} < \infty.
\]
Note that \( h(x) \geq 0 \) is directly Riemann-integrable if \( h(x) \) is continuous, integrable and eventually monotone at both tails.

**Theorem 3.** Suppose that the density function \( p(x) \) of \( X \) exists and is bounded by a directly Riemann-integrable function. Then \( (1.11) \) is valid. Furthermore, suppose that in addition \( E|X|^4 < \infty \). Then for any \( 0 < a \leq b < \infty \),

\[
(1.13) \quad \sum_{n=0}^{\infty} \sup \{|(\partial/\partial \mu)(ER(x, \mu) - r(\mu))| : n \leq x < n + 1, a \leq \mu \leq b\} < \infty.
\]

This theorem is essential for the expansion of variances of nonlinear boundary crossing times in Zhang (1988). The methods in the following sections may also be used to obtain other uniform renewal theorems with varying \( \mu \) or with respect to \( \partial/\partial \mu \) when the needs for such theorems arise. Uniform renewal theorems have been studied by Lai (1976) and Kartashov (1980). Although they considered uniform convergences for larger classes of underlying distribution functions, their results are not delicate enough to meet our needs for nonlinear renewal theory. Also, the convergence in Theorem 3 has not been considered in previous studies. Chow and Lai (1979) investigated the behavior of the stopping times \( T(0, \mu) \) as \( \mu \to 0 \) to obtain results on driftless random walks.

We conclude this section with a proposition which gives an expression for \((\partial/\partial \mu)r(\mu)\).

**Proposition 1.** Let \( r(\mu) \) be defined by \((1.4)\). Suppose that \( EX^2 = \sigma^2 < \infty \) and \( X \) is not a lattice. Then

\[
(\partial/\partial \mu)r(\mu) = (1 - \sigma^2/\mu^2)/2 + \sum_{n=1}^{\infty} P(S_n + \mu n \leq 0).
\]

**2. Proof of Theorem 2.** Since the theorems with \( b = 2a \) can be proved by considering \( X_n/a, n \geq 1 \), we shall assume without loss of generality that the domain of \( \mu \) is \([1, 2]\) in Sections 2 and 3.

Let \( f \) be defined by \((1.6)\). Then, by the Wiener–Hopf factorization [Feller (1971), pages 604–610]

\[
(2.1) \quad f = 1 + (g - 1)U,
\]

where

\[
(2.2) \quad g = g(u, \mu) = E \exp[iu(X + \mu)]
\]

and

\[
(2.3) \quad U = U(u, \mu) = \exp \left[ \sum_{n=1}^{\infty} n^{-1} \int_{-\infty}^{\infty} e^{iux} dP(S_n + n\mu \leq x) \right].
\]
Since
\[ \sum_{n=1}^{\infty} n^{-1} P(S_n + n\mu \leq 0) \]
\[ = \log \left[ EY(\mu)/E(X + \mu) \right] = \log ET(0, \mu) \leq \log ET(0,1), \]
we have by (2.1)
\[ |g(u, \mu) - 1|/ET(0,1) \leq |f(u, \mu) - 1| \leq |g(u, \mu) - 1|ET(0,1), \]
(2.6) \[ g(u, \mu) \neq 1 \text{ when } u \neq 0, \]
and for small \( u, \)
\[ |g - 1| \geq |1 - e^{iu\mu} - |1 - E^{iu\mu x}| \geq \mu |u|/2 \geq |u|/2. \]
Therefore, for any \( M \geq 0, \)
\[ \sup \{|(f(u, \mu) - 1)/u|^{-1} : 0 \leq |u| \leq M, 1 \leq \mu \leq 2 \} < \infty. \]

We shall consider Borel functions on \(( -\infty, \infty) \times [1, 2] \) in the space
\[ (2.9) \ L^* = \left\{ h = h(x, \mu) : \sum_{n=-\infty}^{\infty} \sup_{n \leq x < n+1 \atop 1 \leq \mu \leq 2} |h(x, \mu)| < \infty \right\} \]
and introduce the following notation. For any complex-valued Borel functions \( h, \)
\( h_1 \) and \( h_2 \) on \(( -\infty, \infty) \times [1, 2] \) for which the corresponding operations are applicable, define
\[ Dh = Dh(u, \mu) = \begin{cases} \lim_{u \to 0} Dh(u, \mu), & u \neq 0, \\ D_h h = D_{\mu} h(u, \mu) = -i(\partial h/\partial u)(u, \mu), \\ D_{\mu} h = D_{\mu} h(u, \mu) = (\partial h/\partial \mu)(u, \mu), \end{cases} \]
\[ \hat{h} = \hat{h}(u, \mu) = (h)^\wedge (u, \mu) = \int_{-\infty}^{\infty} e^{iu\mu} h(x, \mu) \, dx, \]
\[ \check{h} = \check{h}(x, \mu) = (h)^\vee (x, \mu) = (2\pi)^{-1} \int_{-\infty}^{\infty} e^{-iu\mu} h(u, \mu) \, du, \]
\[ h_1 \ast h_2 = h_1 \ast h_2(x, \mu) = \int_{-\infty}^{\infty} h_1(x - y, \mu) h_2(y, \mu) \, dy, \]
\[ h^n = h \ast h^{(n-1)}, \quad n \geq 1, \quad h^1 = h, \]
and \( \hat{L}^* = (\hat{h} : h \in L^*) \), where \( L^* \) is defined by (2.9).

By elementary calculus, we immediately have
\[ D \int_{-\infty}^{\infty} e^{iu\mu} dH(x) \]
\[ = \int_{0}^{\infty} e^{iu\mu} (H(\infty) - H(x)) \, dx - \int_{-\infty}^{0} e^{iu\mu} (H(x) - H(-\infty)) \, dx \]
(2.10)
for any monotone function $H(x)$ with $\int_{-\infty}^{\infty} |x| \ dH(x) < \infty$,
\begin{equation}
D_u Dh = DD_u h - D^2 h \quad \text{if} \quad \int_{-\infty}^{\infty} |\hat{h}(x, \mu)| \ dx < \infty
\end{equation}
and
\begin{equation}
\{ h_1, h_2 \} \subset \hat{L}^* \implies \{ h_1 + h_2, h_1 - h_2, h_1 h_2 \} \subset \hat{L}^*.
\end{equation}

Define $f_0 = f$, $g_0 = g$, $U_0 = U$,
\begin{equation}
f_n = Df_{n-1}, \quad g_n = Dg_{n-1}, \quad U_n = DU_{n-1}, \quad n \geq 1,
\end{equation}
and
\begin{equation}
\xi(c, \mu) = \begin{cases} 
ER(c, \mu) - r(\mu) + m^{-1}(\mu) \int_c^\infty \int_x^\infty P\{Y(\mu) \geq y\} \ dy \ dx, \\
0, & c < 0.
\end{cases}
\end{equation}

Then by Definition $f_n(0, \mu) = EY^n(\mu)/n!$, $n \geq 1$.

**Lemma 1.** Let $R$ and $\xi$ be defined by (1.2) and (2.14), respectively. If $E|X|^3 < \infty$, then
\begin{equation}
\xi(u, \mu) = f_2(u, \mu)/(f_1(u, \mu)m_1(\mu)),
\end{equation}
where $f_1$, $f_2$ and $m_1$ are defined by (2.13) and (1.3).

A proof of Lemma 1 may be found in Carlsson (1983). Indeed, (2.15) is the first displayed equation on page 149 of that article. We state without proof the following lemma which follows easily from the argument of Stone (1965), page 334.

**Lemma 2.** For any positive integer $k \geq 2$ there exists a universal constant $A_k < \infty$ such that for any probability distribution function $H(x)$,
\begin{equation}
\int_1^\infty |D^{k-1}_u D^2 h(u)| \ du \leq A_k \int_{-\infty}^{\infty} |x|^k \ dH(x),
\end{equation}
where $h(u) = \int_{-\infty}^{\infty} e^{iux} \ dH(x)$.

Our next lemma is the key to the proofs of Theorems 2 and 3. Since the argument is to be used repeatedly, we define the space $L(k, c)$ for each $c \geq -\infty$ and integer $k \geq 0$ as follows:
\begin{equation}
L(k, c) = \{ h : h \ satisfies\ (2.17)-(2.20) \},
\end{equation}
where
\begin{equation}
(f^n f^j u) (x, \mu) I\{x \geq c\} \in L^*, \quad n \geq 0, 0 \leq j \leq k,
\end{equation}
\begin{equation}
\sup_{(u, \mu)} |D^j_u \hat{h}(u, \mu)| < \infty, \quad j = 0, 1,
\end{equation}
\begin{equation}
\sup_{(u, \mu)} \{ |D^2_u \hat{h}(u, \mu)| : |u| \geq 1, 1 \leq \mu \leq 2 \} < \infty.
\end{equation}
and

\begin{equation}
\sup_{\mu} \int_{-1}^{1} |D_{u}^{2}\hat{h}(u, \mu)| \, du < \infty.
\end{equation}

And we define \( \hat{L}(k, c) = \{ \hat{h}: h \in L(k, c) \} \).

**Lemma 3.** Let \( f_{n}, n \geq 0 \), be defined by (2.13). Suppose that \( E|X|^{3} < \infty \) and (1.11) is satisfied. Then

\begin{equation}
( (f_{2}/f_{1})^{k}\hat{h})^{\gamma}(x, \mu)I\{x \geq c\} \in L^{*}
\end{equation}

for any \( h \in L(k, c), c \geq -\infty \) and \( k \geq 0 \). In particular, \( f_{2} \in \hat{L}(k, -\infty) \) for any \( k \geq 0 \) and

\begin{equation}
f_{2}^{2}/f_{1} \in \hat{L}^{*}.
\end{equation}

**Proof.** Since for \( x \geq 0 \),

\begin{equation}
P \left( \sup_{1 \leq \mu \leq 2} Y(\mu) \geq x + 2 \right) \leq P \left( \max_{j \leq T(0,1)} X_{j} \geq x \right) \leq P(X \geq x)ET(0, 1),
\end{equation}

by (2.10), (2.12) and (2.13) for \( n \geq 0 \) and \( j \geq 0 \),

\begin{equation}
f^{n}f_{1}^{2} = m_{1}f_{2} + (f_{1} - m_{1})D(f^{n}f_{1}) \in \hat{L}^{*}.
\end{equation}

It follows from (2.11) and Lemma 2 that

\begin{equation}
f_{2}^{2} \in L(k, -\infty) \text{ defined by (2.16), } \quad k \geq 0.
\end{equation}

By (1.11) there exists an integer \( m > 0 \) such that

\begin{equation}
\sup_{\mu} \int_{-\infty}^{\infty} \left| f(u, \mu) \right|^{m-2} \, du < \infty.
\end{equation}

Since \( X \) is strongly nonlattice by (1.11) and (2.5),

\begin{equation}
\sup_{(u, \mu)} \{ |g(u, \mu)| = |Ee^{iux}|: |u| \geq 1, \mu \} < 1
\end{equation}

and by (2.5)--(2.8)

\begin{equation}
\sup_{(u, \mu)} |f_{1}(u, \mu)(1 + |u|)^{-1} < \infty.
\end{equation}

By algebra

\begin{equation}
f_{2}/f_{1} = (m_{1} - f_{1})/(1 - f) = (1 + f + \cdots + f^{m-1})(m_{1} - f_{1}) + f^{m}f_{2}/f_{1}.
\end{equation}

It follows that

\begin{equation}
(f_{2}/f_{1})^{k}\hat{h} = (1 + f + \cdots + f^{m-1})^{k}(m_{1} - f_{1})^{k}\hat{h} + f^{m}Q_{0}\hat{h},
\end{equation}

where \( Q_{0} \) is a polynomial of \( f, f_{1} \) and \( f_{2}/f_{1} \). By (2.17) and (2.12)

\begin{equation}
\left[ (1 + f + \cdots + f^{m-1})^{k}(m_{1} - f_{1})^{k}\hat{h} \right]^{\gamma}(x, \mu)I\{x \geq c\} \in L^{*}.
\end{equation}
To bound \((f^m Q_0 \hat{h})^\vee\), we consider

\[(2.27) \quad D^2_u (f^m Q_0 \hat{h}) = f^{m-2} (Q_1 D^2_u \hat{h} + Q_2 (D^a f_3) / f_1 + Q_3),\]

where \(Q_1\), \(Q_2\) and \(Q_3\) are polynomials of \(D^j \hat{h}\), \(j = 0, 1, D^j f_3, 0 \leq n \leq 2, 0 \leq j \leq 2, j + n \leq 3\), and \((D^j f_n) / f_1\), \(n = 1, 2\), \(0 \leq j \leq 2, n + j \leq 3\). Since \(E|X|^3 < \infty\), by (2.10), (2.11), (2.21) and (2.24)

\[
\sup_{(u, \mu)} |(D^j f_n) / f_1| < \infty \quad \text{for all } n = 1, 2, 0 \leq j \leq 2, n + j \leq 3,
\]

so that by (2.18), (2.10) and (2.21)

\[
\sup_{(u, \mu)} |Q_n(u, \mu)| < \infty, \quad 0 \leq n \leq 3.
\]

By (2.19), (2.11) and (2.24)

\[
\sup_{(u, \mu)} |D^2_u \hat{h}(u, \mu)| + |f^{-1}(u, \mu) D^a f_3(u, \mu)|: |u| \geq 1, 1 \leq \mu \leq 2 \rangle \right) < \infty.
\]

By (2.20), (2.24) and Lemma 2

\[
\sup_{\mu \in (-1, 1)} \int_{-\infty}^{\infty} |D^2_u \hat{h}(u, \mu)| + |f^{-1}(u, \mu) D^a f_3(u, \mu)|\, du < \infty.
\]

Therefore, by (2.27) and (2.23)

\[
\sup_{\mu \in (-\infty, \infty)} \int_{-\infty}^{\infty} |D^2_u (f^m Q_0 \hat{h})(u, \mu)|\, du < \infty.
\]

By the inversion formula for Fourier transformations

\[
(1 + x^2)(f^m Q_0 \hat{h})^\vee(x, \mu) = (2\pi)^{-1} \int_{-\infty}^{\infty} e^{-ixu} (1 + D^2_u) (f^m Q_0 \hat{h})(u, \mu)\, du.
\]

It follows that

\[
\sup_{(x, \mu)} \left| (1 + x^2)(f^m Q_0 \hat{h})^\vee(x, \mu) \right|
\]

\[
\leq (2\pi)^{-1} \left[ \sup_{\mu \in (-\infty, \infty)} \int_{-\infty}^{\infty} |f|^m du \right] \sup_{(u, \mu)} |Q_0 \hat{h}| + \sup_{\mu \in (-\infty, \infty)} \int_{-\infty}^{\infty} |D^2_u (f^m Q_0 \hat{h})|\, du
\]

\[
< \infty.
\]

Hence, \(f^m Q_0 \hat{h} \in L^*\) and by (2.25) and (2.26)

\[
((f_3 / f_1)^k \hat{h})^\vee(x, \mu) I\{x \geq c\} \in L^*.
\]

\[\square\]

**Proof of Theorem 2.** By (2.10)

\[
(D^n f)(x, \mu) = P\{Y_1 + \cdots + Y_n > x\} I\{x \geq 0\},
\]

\[
(D^2 f)(x, \mu) = \int_{x}^{\infty} P\{Y_1 + \cdots + Y_n > y\} dy I\{x \geq 0\}.
\]

It follows from (2.21) that there exists a function \(M(x)\) not depending on \(\mu\) such
that $M(x) = 0$ for $x < 0$, $M(x)$ is decreasing for $x \geq 0$, $\int_0^\infty M(x) \, dx < \infty$ and

$$
(2.29) \quad |(D^j f^n)\gamma(x, \mu)| \leq M(x), \quad j = 1, 2, 1 \leq n \leq m.
$$

Since

$$
(2.30) \quad f_2 f^n = f_2 + (f_1 - m_1) Df^n = D^2 f + (D \mu - m_1) Df^n,
$$

$(1 + f + \cdots + f^{m-1})(m_1 - f_1)f_2$ is a polynomial of $Df^n$, $j = 1, 2$, $1 \leq n \leq m$, so that by (2.29) and the fact that $M * M(x) \leq 2M(x/2)\int_0^\infty M(y) \, dy$

$$
\int_0^\infty \sup \left\{ |((1 + f + \cdots + f^{m-1})(m_1 - f_1)f_2)^\gamma(y, \mu)| : y \geq x, 1 \leq \mu \leq 2 \right\} \, dx < \infty.
$$

Replace $\hat{h}$ by $\hat{f}_2$ and take $k = 1$ in (2.25) and (2.28), and we have by (2.22)

$$
(2.31) \quad \int_0^\infty \sup \left\{ |(\hat{f}_2^2/\hat{f}_1)^\gamma(y, \mu)| : y \geq x, 1 \leq \mu \leq 2 \right\} \, dx < \infty.
$$

The proof is complete since (1.12) follows from (2.31), Lemma 1, (2.21), (2.14) and (2.29). \(\square\)

3. **Proof of Theorem 3.** We split the proof into several lemmas.

**Lemma 4.** Let $p(x)$ be the density function of $X$ and define

$$
(3.1) \quad q = q(x, \mu) = \sum_{n=1}^\infty p_{\mu n} (x - n\mu) I\{x \leq 0\}.
$$

Suppose that $p(x)$ is bounded by a directly Riemann-integrable function. Then

$$
\sup_{(x, \mu)} q(x, \mu) < \infty.
$$

**Proof.** Let $M > 0$ be an upper bound for $p(x)$ and $\sum_{n=1}^\infty P\{k < S_n + n\mu \leq k + 1\}$ for all $1 \leq \mu \leq 2$ and $k$. Theorem 1 ensures such an upper bound $M \leq \infty$. Given $x \leq 0$,

$$
q(x, \mu) = p(x - \mu) + \sum_{n=1}^\infty \int_{-\infty}^\infty p(x - y - \mu) \, dP(S_n + n\mu \leq y)
$$

$$
\leq M + \left[ \sup_k \sum_{n=1}^\infty P(k \leq S_n + n\mu < k + 1) \right] \sum_{k \leq y < k+1} \sup \left\{ p(x - y - \mu) \right\}
$$

$$
\leq M \left[ 1 + 2 \sum_{k \leq y < k+1} \sup \left\{ p(y) \right\} \right] < \infty. \quad \square
$$

**Lemma 5.** Let $h$ and $h_1$ be two Borel functions on $(-\infty, \infty) \times [1, 2]$ such that $h(x, \mu)$ vanishes on $(0, \infty) \times [1, 2]$. Suppose that $\sup_{(x, \mu)} |h(x, \mu)| < \infty$ and $\int_{-\infty}^\infty |h_1(y, \mu)| \, dy I\{y \geq 0\} \in L^2$. Then

$$
\int_{-\infty}^\infty |h_1(y, \mu)| \, dy I\{y \geq 0\} \in L^2.
$$
PROOF. Clearly,
\[ |h \ast h_1(x, \mu)| = \left| \int_{-\infty}^{\infty} h(x - y, \mu) h_1(y, \mu) \, dy \right| \]
\[ \leq \int_{x}^{\infty} |h(x - y, \mu) h_1(y, \mu)| \, dy \]
\[ \leq \left[ \sup_{(y, \mu)} |h(y, \mu)| \right] \int_{x}^{\infty} |h_1(y, \mu)| \, dy \in L^*. \] □

**Lemma 6.** If \( E|X|^3 < \infty \), then
\[ D_\mu f_1 = f_1(iu \hat{q} - q(0, \mu)) + gU, \]
\[ D_\mu f_2 = f_1 \hat{q} - f_2 q(0, \mu) + f_1 + U_1 \]
and
\[ D_\mu f_3 = f_2 \hat{q}(0, \mu) + f_1 D \hat{q} - f_2 q(0, \mu) + f_2 + U_2, \]
where \( f_n, g, U_n, n \geq 0 \), and \( q \) are defined by (2.13) and (3.1).

**Proof.** By (2.1)–(2.3)
\[ D_\mu g = iug, \quad D_\mu g_1 = g, \quad f_1 = g_1 U \quad \text{and} \quad D_\mu f_1 = g_1 D_\mu U + UD_\mu g_1. \]
Since by (2.3)
\[ D_\mu U = UD_\mu \sum_{n=1}^{\infty} n^{-1} \int_{-\infty}^{0} e^{iux} p^n(x - n\mu) \, dx = U(iu \hat{q} - q(0, \mu)), \]
we have
\[ D_\mu f_1 = f_1(iu \hat{q} - q(0, \mu)) + gU. \]
And the expressions for \( D_\mu f_n \) may be derived from
\[ D_\mu f_n = D_\mu Df_{n-1} = DD_\mu f_{n-1}. \] □

**Proof of Theorem 3.** (i) Let us first prove that (1.11) holds. By (2.1)–(2.3)
\[ (\log U)^\gamma(x, \mu) = \sum_{n=1}^{\infty} n^{-1} p^n(x - n\mu) I(x \leq 0) \]
\[ \leq q(x, \mu) \leq M < \infty \]
and
\[ ((\log U)^\gamma)^n(x, \mu) \leq M \left( \int_{-\infty}^{\infty} (\log U)^\gamma(y, \mu) \, dy \right)^{n-1} \]
\[ \leq M \left( \sum_{j=1}^{\infty} j^{-1} P(S_j + j \leq 0) \right)^{n-1} < \infty. \]
By (2.4)
\[(U - 1)^{\gamma}(x, \mu) \leq \sum_{n=1}^{\infty} (n!)^{-1}((\log U)^{\gamma})^n(x, \mu) \leq \text{MET}(0, 1) < \infty.\]

(3.2)

It follows from (2.1) that \(f = g + g(U - 1) - (U - 1)\) and
\[
\hat{f}(x, \mu) = \left[p(x - \mu) + \int_{-\infty}^{\infty} (U - 1)^{\gamma}(x - y, \mu)p(y - \mu)\,dy\right]I(x \geq 0).
\]

By Lemma 5, (3.2), (2.10) and (2.21)
\[
(3.3) \quad \hat{f}_n \in L^*, \quad \text{if } E|X|^n < \infty, \quad n \geq 1.
\]

Therefore, (1.11) is satisfied with \(k = 2\) by Parseval’s identity and the uniform boundedness of \(\hat{f}(x, \mu)\).

(ii) Since by (2.10), (2.14) and Lemma 1 the Fourier transformation of \(ER(x, \mu) - r(\mu)\) on the positive half-axis is \(f_2^2/(\hat{f}_1 m_1) - f_2/m_1\), we shall consider \((f_2/\hat{f}_1)D_2\hat{f}_2\), \((f_2/\hat{f}_1)^2D_2\hat{f}_2\), and \(D_2\hat{f}_2\). By (2.1), \(\hat{f}_1 = g\hat{U}\) and \(\hat{U}_1 = D\hat{U}\). It follows from Lemma 6 that
\[
(3.4) \quad (f_2/\hat{f}_1)D_2\hat{f}_2 = f_2(\hat{q} + 1) + (\mu^{-1} - q(0, \mu))(f_2^2/\hat{f}_1 - \mu^{-1}(f_2/\hat{f}_1)g\hat{U}).
\]

Since \(E|X|^3 < \infty\) and \(U - 1\) is the Fourier transformation of a nonnegative integrable function,
\[
\sup_{(u, \mu)} |D_2^2U(u, \mu)| = \sup_{\mu} |D_2^2U(0, \mu)|
\]
\[
(3.5) \quad = \sup_{\mu} 2U_2(0, \mu)
\]
\[
= \sup_{\mu} 2\mu^{-1}(Df_2 - D(Ug_2))(0, \mu) < \infty.
\]

By (3.2), Lemma 5, (2.10), (2.21), (2.29) and (3.5)
\[
(3.6) \quad U - 1 \in \hat{L}(k, 0), \quad k = 1, 2.
\]

Since \(g\hat{U} = g\hat{U}(U - 1) + g_2\), by (3.5), (3.6) and Lemma 2
\[
(3.7) \quad g\hat{U} \in \hat{L}(k, 0), \quad k = 1, 2.
\]

By (3.3) and Lemmas 4 and 5, \((f_2(\hat{q} + 1))^*I(x \geq 0) \in L^*.\) Therefore, we have by (3.4), Lemma 3 and (3.7)
\[
(3.8) \quad (\hat{f}_2/\hat{f}_1)D_2\hat{f}_2 \in L^*.
\]

For \((f_2/\hat{f}_1)^2D_2\hat{f}_2\) we have by Lemma 6 and the definition of \(D,\)
\[
(3.9) \quad (f_2/\hat{f}_1)^2D_2\hat{f}_2 = f_2\hat{q} - m_1(f_2/\hat{f}_1)\hat{q} - q(0, \mu)f_2^2/\hat{f}_1 + (f_2/\hat{f}_1)^2\hat{g}U.
\]

Since \(gU = f + (U - 1),\)
\[
(3.10) \quad gU \in \hat{L}(k, 0), \quad k = 1, 2, \quad \text{by (3.3) and (3.6).}
It follows from Lemmas 4, 5, 3 and (3.10) that
\begin{equation}
( f_2 \hat{q} - q(0, \mu) f_2^2 / f_1 + ( f_2 / f_1 )^2 g U )^\tau ( x, \mu ) I\{ x \geq 0 \} \in L^*.
\end{equation}
Let us use the assumption that $E|X|^4 < \infty$. By Stone (1965)
\begin{equation}
\int_{-\infty}^{0} |x| \sum_{n=1}^{\infty} P(S_n + n \leq x) \, dx < \infty,
\end{equation}
which implies that
\begin{equation}
| D_2^2 \hat{q}(u, \mu) | \leq 2 \int_{-\infty}^{0} |x| \sum_{n=1}^{\infty} P(S_n + n \leq x) \, dx < \infty.
\end{equation}
By Lemmas 3, 4, 5, (3.3) and (3.13)
\begin{equation}
(( f_2 / f_1 ) \hat{q})^\tau (x, \mu) I\{ x \geq 0 \} \in L^*.
\end{equation}
By (3.9), (3.11), (3.14) and (2.12)
\begin{equation}
(( f_2 / f_1 )^2 D_2 f_1)^\tau (x, \mu) I\{ x \geq 0 \} \in L^*.
\end{equation}
Now let us consider $D_\mu m_1$. By Lemma 6
\begin{equation}
D_\mu m_1 = D_\mu f_1(0, \mu) = -q(0, \mu) m_1 + U(0, \mu),
\end{equation}
which implies that
\begin{equation}
\sup_{\mu} |D_\mu m_1(\mu)| < \infty.
\end{equation}
Since $(m_1(\mu))^{-1}$ is bounded, by Lemma 1, (3.8), (3.15), (3.16), Theorem 2 and (2.12)
\begin{equation}
(D_\mu \hat{q})^\tau (x, \mu) I\{ x \geq 0 \} = (D_\mu ( f_2^2 / (f_1 m_1)))^\tau (x, \mu) I\{ x \geq 0 \} \in L^*.
\end{equation}
Finally, let us consider $D_\mu f_3$. Again by Lemma 6
\begin{equation}
D_\mu f_3 = f_2 \hat{q}(0, \mu) + f_1 D\hat{q} - f_2 q(0, \mu) + f_2 + U_2.
\end{equation}
Since
\begin{equation}
(U_2)^\tau (x, \mu) I\{ x \geq 0 \} = 0
\end{equation}
and
\begin{equation}
(D\hat{q})^\tau (x, \mu) = \sum_{n=1}^{\infty} P(S_n + n \mu \leq x) I\{ x \leq 0 \},
\end{equation}
we have by (3.3) and Lemmas 5 and 4
\begin{equation}
(D_\mu ( f_2 / m_1))^\tau (x, \mu) I\{ x \geq 0 \} \in L^*.
\end{equation}
Let $h_0 = I\{ x \geq 0 \}$. It follows from (2.14), (2.10), (3.17) and (3.18) that
\begin{equation}
(D_\mu (((ER - r) h_0) ^\tau ))^\tau h_0 = (D_\mu ( f_2^2 / (f_1 m_1) - f_2 / m_1))^\tau h_0 \in L^*.
\end{equation}
Let \( h_1(x) = x^2 \exp[-|x|] \) and \( h_2 \) be the right-hand side of (3.19). Then
\[
(h_2 h_1)^\wedge = (2\pi) \left( D_\mu \left[ \left( (E R - r) h_0 \right)^\wedge \right] \right) \ast \left( (h_0 h_1)^\wedge \right)
= (2\pi) D_\mu \left[ \left( (E R - r) h_0 \right)^\wedge \right] \ast \left( h_0 h_1 \right)^\wedge
= D_\mu \left[ \left( (E R - r) h_0 h_1 \right)^\wedge \right].
\]
Hence, by Lebesgue's dominated convergence theorem
\[
(ER - r) h_0 = \left( 1/h_1 \right) \left( (D_\mu)^{-1} \left[ \left( h_2 h_1 \right)^\wedge \right] \right)^\wedge
= \left( 1/h_1 \right) (D_\mu)^{-1} (h_2 h_1) \quad \text{(exchange integral } D_\mu^{-1} \text{ and } ^\wedge) \\
= (D_\mu)^{-1} h_2
\]
and the proof is complete by (3.19). \( \Box \)

**Proof of Proposition 1.** By (2.1)—(2.3) \( f_1 = g_1U \) and \( f_2 = g_2U + \mu U_1 \). Since \( f_1(0, \mu) = EY(\mu), \ f_2(0, \mu) = EY(\mu)^2/2, \ g_1(0, \mu) = \mu \) and \( g_2(0, \mu) = (E X^2 + \mu^2)/2 \),
\[
r(\mu) = (2\mu)^{-1} (\sigma^2 + \mu^2) - \sum_{n=1}^{\infty} n^{-1} E(S_n + n\mu)^-. 
\]
It follows that
\[
D_\mu r(\mu) = \left( 1 - \sigma^2 / \mu^2 \right) / 2 - \sum_{n=1}^{\infty} n^{-1} D_\mu \int_{-\mu n}^{\infty} P(S_n \leq -x) \, dx \\
= \left( 1 - \sigma^2 / \mu^2 \right) / 2 + \sum_{n=1}^{\infty} P(S_n + \mu n \leq 0). \quad \Box
\]

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