5. Bayes Rule

Suppose $C_1, C_2, \dots$ are mutually exclusive and exhaustive cases (hypotheses) and E is some event (evidence) that has been observed. We have some prior "belief" on each hypothesis, which we write $P[C_1], P[C_2], \dots$ and we want to update our beliefs in the face of the evidence in E. By standard conditional probability calculations, we have

$$P[C_i \mid E] = \frac{P[C_i]P[E \mid C_i]}{\sum_j P[C_j]P[E \mid C_j]}, \quad \text{for each } j \geq 1. \tag{10}$$

(Note that if there are only two cases then

$$\frac{P[C \mid E]}{P[C^c \mid E]} = \frac{P[C]}{P[C^c]} \times \frac{P[E \mid C]}{P[E \mid C^c]}.$$

5.1. Bayes theorem and the law.

Bayesian reasoning naturally arises in connection with legal proceedings. For example, in a criminal trial, some crime was committed, for which they may be various hypotheses as to what happened and who is responsible. Evidence is collected in an attempt to deduce which of the hypotheses actually occurred. We assume that the hypotheses are exhaustive in the sense that they contain all *reasonable* scenarios and, most importantly, one of the hypotheses is correct. (This is not always the case.) In the language of Bayes rule (10): of all possible outcomes, there are several hypotheses that might be feasible (say, $C_1, \dots, C_k$) and we collect evidence to discern which hypothesis is most likely. Before seeing the evidence, we might have a belief (rather, a "hunch") as to which of the hypotheses are more likely. After viewing the evidence, we update our beliefs (perhaps, according to Bayes rule).

The use of Bayesian reasoning in criminal trials is controversial. For example, the case of *R v. Adams* was a landmark case in which a prominent statistician Peter Donnelly gave expert testimony explaining Bayes theorem and how it applied to the case. The problem is that Bayes theorem confuses many jurors. Moreover, the conclusion of "one in a million" can be construed as "highly unlikely" by some and "rather plausible" by others. Wikipedia contains more information about this case.

We demonstrate the utility of Bayes theorem using data from the O.J. Simpson trial, regarded by many as "the trial of the century."

5.2. O.J. Simpson trial.

O.J. Simpson was an American football legend. After his Hall of Fame playing career, he maintained his celebrity status by acting in movies, commercials, and working as an analyst on NBC Football broadcasts. For these, and many other reasons, the murder of his wife, Nicole Simpson, and a friend, Ron Goldman, in June 1994 dominated the news for more than a year. The trial, which lasted nearly a year, was broadcast live on television.

There were several key pieces of evidence linking Simpson to the crime. Among the mountain of evidence against Simpson was the presence of a bloody glove behind Simpson's Rockingham estate, which was found by LAPD detective Mark Fuhrman. The glove matched one found at the scene of the crime and videos from NBC football broadcast showed Simpson wearing the same type of gloves in the past. Aside from this, blood in Simpson's car and house matched blood at the scene and, as the prosecution pointed out at trial, Simpson had a history of abuse toward his wife. Despite the overwhelming evidence, Simpson was ultimately found not guilty.

He we focus on a particular quote from Defense lawyer Alan Dershowitz, which was used to refute the relevance of Simpson's previous abuse. During trial, Dershowitz cited a statistic that

"Only one in a thousand abusive husbands eventually murder their wives."

Dershowitz was suggesting that the chance that Simpson was guilty of the crime was quite small, only 1 in 1000; therefore, no reasonable person could find him guilty based on this evidence alone. Is this a valid argument?

As it turns out, Dershowitz is ignoring a crucial piece of evidence, namely, that a murder has actually occurred. To quantify the deceit of Dershowitz's claim, we appeal to Bayes rule. Let G be the event that a husband murdered his wife, B the event that a husband abuses his wife, and M be the event that the wife is murdered. According to Dershowitz, $P(G \mid B) = 1/1000$, and so we will take this as given. However, in this case, we have the additional information that the wife was murdered, and so we want to calculate $P(G \mid B, M)$. By Bayes Rule:

$$\begin{aligned} P(G \mid B, M) &= \frac{P(M \cap G \cap B)}{P(M \cap B)} \\ &= \frac{P(M \mid G, B)P(G \mid B)P(B)}{P(M \mid B)P(B)} \\ &= \frac{P(M \mid G, B)P(G \mid B)}{P(M \mid B)} \\ &= \frac{P(M \mid G, B)P(G \mid B)}{P(M \mid B, G)P(G \mid B) + P(M \mid B, G^c)P(G^c \mid B)}. \end{aligned}$$

To estimate this probability, we collect the following data.

- $P(M \mid G) = P(M \mid G, B) = 1$
- $P(M \mid G^c)$: in 1994, 5000 women were murdered, 1500 by their husband. Assuming a population of 100 million women, we have
$$P(M \mid G^c) = \frac{3500}{100 \times 10^6} \approx \frac{1}{30,000}.$$
- $P(G \mid B) = 1/1000$.
- $P(G^c \mid B) = 999/1000$.

Applying Bayes rule, we have

$$P(G \mid M, B) = \frac{1 \times 1/1000}{1 \times 1/1000 + 1/30,000 \times 999/1000} = \frac{30000}{30,999} \approx 0.97.$$

Years later, Simpson wrote a book entitled *If I Did It* in which he described how he would have committed the murder, **if** he were the one who actually did it. Quite coincidentally, Simpson would have done things pretty much exactly the same way. To this day, Simpson continues to actively search for the real killers.