STAT 592: HOMEWORK 3

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1. Homework 3

Problem 1.1. *Suppose Y is a standard Cauchy random variable with density*

$$f(y) = \frac{1}{\pi}\frac{1}{1+y^2} I_{(-\infty,\infty)}(y).$$

(i) *Compute the first and third quartiles of Y.*
(ii) *Show that $P(Y \geq y \sim 1/(\pi y)$ as $y \to \infty$, where $a_n \sim b_n$ as $n \to \infty$ indicates that $a_n/b_n \to 1$ as $n \to \infty$.*

Problem 1.2. *Let X have the Pareto distribution with parameter (k, α) with density*

$$f(x) = \frac{\alpha k^\alpha}{x^{\alpha+1}} I_{[k,\infty)}(x).$$

Find the density of $Y = k/X$.

Problem 1.3. *Let U_1, U_2 be i.i.d. Uniform$[0,1]$ and define $X_1 = \sqrt{-2\log U_1}\cos(2\pi U_2)$ and $X_2 = \sqrt{-2\log U_1}\sin(2\pi U_2)$. Find the density of (X_1, X_2).*

Problem 1.4. *Let X_1, X_2 be independent standard normal random variables and let $Y = X_1^2 + X_2^2$ and $Z = X_1/X_2$.*

(i) *Find the joint density of Y and Z and find the marginal densities of Y and Z.*
(ii) *Use QQ plots on simulated data to demonstrate that your marginal densities are correct. Show the plots and the work you have done to construct them. (Hint: Use QQ plot capability in R).*

Problem 1.5. *Let $X_1, \ldots, X_n$ be independent random variables from the exponential distribution with rate $\lambda > 0$, that is, $X_1, \ldots, X_n$ have density $f(x) = \lambda e^{-\lambda x} I_{(0,\infty)}(x)$. Find the density of $R = X_{(n)} - X_{(1)}$, where*

$$X_{(1)} < \cdots < X_{(n)}$$

are the order statistics.

Problem 1.6. *Let $X \sim N(0,1)$ and let $g : \mathbb{R} \to \mathbb{R}$ denote a differentiable function such that $E|g'(X)| < \infty$. Show that*

$$E(g'(X)) = E(Xg(X)).$$

Problem 1.7. *Find the expectation of the following:*

(i) *The negative binomial distribution with parameters $r \geq 1$ and $0 < p < 1$. That is,*

$$P(X = k) = \binom{k+r-1}{k}(1-p)^r p^k, \quad k = 0, 1, \ldots.$$

Problem 1.8. *Let X be a random variable.*

(i) *Show that X is integrable if and only if $\sum_{n\geq 1} \mathbb{P}(|X| \geq n) < \infty$.*

Now let $X \geq 0$ be a nonnegative random variable.

(ii) *Show that there exists a function $f : [0, \infty) \to [0, \infty)$ that is increasing such that $f(0) = 0$, $f(x) \to \infty$ as $x \to \infty$, and $f(X)$ is integrable.*

Problem 1.9. *Show that for arbitrary random variables X_1 and X_2 with variances σ_1^2 and σ_2^2 and correlation coefficient ρ, we have for arbitrary $k > 0$,*

$$P(A_1 \cup A_2) \leq \frac{1 + \sqrt{1-\rho^2}}{k^2},$$

where $A_i = \{|X_i - E(X_i)| \geq k\sigma_i\}$ for $i = 1, 2$.

Problem 1.10. *Let X be a k-dimensional random vector distributed according to a multinomial distribution, i.e., there exists a positive integer $n \geq 1$ and a probability vector $p = (p_1, \ldots, p_k)$ such that*

$$\mathbb{P}(X_1 = n_1, \ldots, X_k = n_k) = \frac{n!}{\prod n_i!} \prod_{p_i^{n_i}}, \quad \sum_{i=1}^{k} n_i = n.$$

(i) *Find the marginal distribution of X_i.*

(ii) *Show that*

$$\mathbb{P}(X_1 = n_1, \ldots, X_k = n_k) \approx (2\pi n)^{\frac{1-k}{2}} e^{-y^2/2} \prod_{i=1}^{k} p_i^{-1/2},$$

for large n, where

$$y^2 = \sum_{i=1}^{k} \frac{(n_i - np_i)^2}{np_i}.$$

Give an interpretation of the approximation.