STAT 592: HOMEWORK 2

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1. Homework 2

Problem 1.1. *For random variables X and Y, $X =_{\mathcal{L}} Y$ means*

$$P_X(X \in B) = P_Y(Y \in B) \quad \text{for all } B \in \mathcal{B}_{\mathbb{R}}, \tag{1}$$

where $\mathcal{B}_{\mathbb{R}}$ is the Borel σ-field on $\mathbb{R}$. For our purposes, it is sufficient to establish (1) for $B = (-\infty, x]$, $x \in \mathbb{R}$.

(i) *Suppose X and Y are random variables and $X =_{\mathcal{L}} Y$. Show that $T(X) =_{\mathcal{L}} T(Y)$ for every transformation $T : \mathbb{R} \to \mathbb{R}$ such that $T^{-1}(B) \in \mathcal{B}$ for every $B \in \mathcal{B}$.*
(ii) *Suppose X and Y are defined on the same probability space $(\Omega, \mathcal{F}, P)$. Show that $P(X = Y) = 1$ implies $X =_{\mathcal{L}} Y$.*

Problem 1.2. *Let Z be a standard normal random variable with density $\phi_Z(z) = (2\pi)^{-1/2} e^{-z^2/2} \mathbf{1}_{(-\infty,\infty)}(z)$.*

(i) *Show that $P_Z(Z \geq z) = (1 + o(1))\phi_Z(z)/z$, where $o(1)$ denotes a quantity that goes to 0 as $z \to \infty$.*
(ii) *Let q_α be the $(1-\alpha)$ quantile of Z. Show that*

$$q_\alpha = \sqrt{2\log(1/\alpha) - \log(\log(1/\alpha)) - \log(4\pi) + o(1)}$$

as $\alpha \downarrow 0$, where $o(1)$ denotes a quantity that goes to 0 as $\alpha \downarrow 0$.

Problem 1.3. *Let $\lambda > 0$, $n \to \infty$, and $p_n \to 0$ such that $np_n \to \lambda$. Show that*

$$\lim_{n\to\infty} \binom{n}{k} p_n^k (1 - p_n)^{n-k} = \frac{\lambda^k e^{-\lambda}}{k!}, \quad k = 0, 1, \ldots.$$

Problem 1.4. *Let X, Y be independent Poisson random variables with intensities $\lambda_1, \lambda_2 > 0$, respectively.*

(i) *Show that $X + Y \sim \text{Poisson}(\lambda_1 + \lambda_2)$.*
(ii) *Compute the conditional distribution of X given $X + Y$.*

Problem 1.5. X stochastically dominates Y, *written $Y \leq X$, if $P(X \geq t) \geq P(Y \geq t)$ for all $t \in (-\infty, \infty)$. Show that if $Y \leq X$ then there exist random variables X' and Y', defined on the same probability space, such that $X' \sim X$, $Y' \sim Y$, and $P(Y' \leq X') = 1$.*