You have until 4:40 PM to finish this exam.

1. I buy a carton of toothbrushes. The carton has 20 toothbrushes. Three of them are blue. I pull out four toothbrushes, one for each member of my family. Let \( X \) represent the number of toothbrushes that my family uses that are blue.
   a. (8 pts) Is the distribution of \( X \) continuous or discrete?
   **Discrete.**
   b. (8 pts) What is the expectation of the distribution of \( X \)?
   -4 if you get wrong expectation because distribution is wrong. No penalty if you give the proper expectation for Binomial.
   \( E[X] = \frac{3 \times 4}{20} = 0.6 \).
   c. (8 pts) Calculate \( P(X \geq 2) \).
   -6 if you use the binomial. -3 if you do 1-complement but omit 0. -2 if you do 1-complement but count 2 in the complement. -4 if hypergeometric is catastrophically wrong.
   \[
P(X \geq 2) = P(X = 2) + P(X = 3) = \binom{3}{2} \frac{17}{2} \binom{20}{4} + \binom{3}{3} \frac{17}{1} \binom{20}{4} = \]
   \[
   \frac{3 \times 17 \times 8}{5 \times 19 \times 3 \times 17} + \frac{17}{5 \times 19 \times 3 \times 17} = 425/4845 = 5/57 = 0.0877 .
   \]
   **Total for this question: 24.**

2. Peanut butter jars are labeled as containing 40 oz. of peanut butter, but the actual weight is variable and exceeds 40 oz. Assume that the amount by which the jar is overfilled is uniformly distributed between 0 and 1 oz. Assume that the amounts by which the jars are overfilled are independent. Choose three jars, and let \( X \) be the median amount by which the jars were overfilled.
   a. (8 pts) Is the distribution of \( X \) continuous or discrete?
   **Continuous.**
   b. (8 pts) What is the expectation and variance of the distribution of \( X \)?
   -3 if moments are from uniform but ignore order statistic aspect
   \( E[X] = 0.5 \); by symmetry, and note that \( X \sim Beta(2, 2) \). Then
   \( E[X^2] = B(4, 2)/B(2, 2) = \frac{\Gamma(4)\Gamma(2)}{\Gamma(2)\Gamma(4)} = \frac{3!3!}{5!} = 3/10 \), and \( Var[X] = 3/10 - 1/4 = 1/20 \).
   c. (8 pts) Calculate \( P(X \leq .7) \).
   (8 pts)-4 if CDF is from uniform but ignores order statistic aspect
   -2 if uses normal approximation, -6 if nowhere close
   \[
P(X \leq .7) = \int_0^{.7} 7u(1-u)6 \ du = 3u^2 - 2u^3 \bigg|_0^{.7} = 3 \times .49 - 2 \times .343 = .784 .
   \]
   **Total for this question: 24.**

3. Let \( X \) be the lifetime of a leaf blower, as measured in hours. Experiments show that the density of \( X \) is well-approximated by \( (.01)^0.5 \exp(-.01x)^{0.5} \).
   a. What name is given to this distribution?
   (6 pts)-4 for Gamma or Exponential Weibull.
   b. What is the expectation and variance of the distribution of \( X \)?
Basic Probability and Statistics - Fall, 2010

(10 pts) 3 of mean and variance are from a distribution other than the uniform, -2 for each answer with a Gamma unevaluated, -1 for each answer with factorial unevaluated, -1 for each variance or probability negative, -1 for each Gamma function assumed to be linear or multiplicative in arguments.

\[ E[X] = (0.01)^{-1} \Gamma(1 + 1/0.5) = (0.01)^{-1} \Gamma(3) = 200, \text{ and} \]
\[ E[X^2] = (0.01)^{-2} \Gamma(1 + 2/0.5) = \Gamma(5) = 240000. \text{ Hence Var}[X] = 240000 - 40000 = 200000. \]

(c) What is the median of the distribution of \( X \)?

\[ \text{Median satisfies } 1 - \exp(-(0.01x)^{0.5}) = 0.5, \\]
\[ \exp(-(0.01x)^{0.5}) = 0.5, \quad -(0.01x)^{0.5} = -\log(0.5), \quad (0.01x)^{0.5} = \log(2), \quad 0.01x = \log(2)^2, \]
\[ x = 100\log(2)^2 = 100 \times 0.69135^2 = 48.05. \]

(d) Suppose 400 such leaf blowers were tested. Approximate the probability that the sample average of the times until failure is less than 210 hours.

\[ P(\overline{X} \leq 210) = \Phi((210 - 200)/\sqrt{200000/400}) = \Phi(0.447) = 0.673. \]

Total for this question: 32.

4. Let \( X_1, X_2, \ldots, X_n \) be random variables with the density \( \frac{1}{1+x^2} \) for \( x \in (-\infty, \infty) \); suppose further that these random variables are independent.

(a) What is the median of each of the \( X_j \) ?

\[ \text{(10 pts)} \text{Set integral of density to 0 -7; solve density=1/2 -5; set integral of density to 1/2 and stop -3 0.} \]

(b) Do you expect that the mean of these random variables will be approximately normal for large \( n \)? Why or why not?

\[ \text{(10 pts)} \text{Result for median -6 No, since the variance is not finite.} \]