1. Question 2.5.8. skip e and f.

a. (a) Set $\int f(x,y) \, dx \, dy = 1$. Then, $\int_0^5 \int_0^5 A(20 - x - 2y) \, dx \, dy = 1$. So

$$\int_0^5 \int_0^5 A(20 - x - 2y) \, dx \, dy = \int_0^5 A(20x - \frac{x^2}{2} - 2xy) \bigg|_0^5 \, dy$$

$$= \int_0^5 A(100 - \frac{25}{2} - 10y) \, dy$$

$$= A(100y - \frac{25}{2}y - 5y^2) \bigg|_0^5$$

$$= A(500 - 125 - 125)$$

$$= A \frac{625}{2}$$

$$= 1$$

$$A = \frac{2}{625}$$

b.

$$P(1 \leq X \leq 2, 2 \leq Y \leq 3) = \int_3^2 \int_1^2 \frac{2}{625} (20 - x - 2y) \, dx \, dy$$

$$= \int_3^2 \frac{2}{625} (20x - \frac{x^2}{2} - 2xy) \bigg|_1^2$$

$$= \frac{1}{625} \int_3^2 (37 - 4y) \, dy$$

$$= \frac{1}{625} [(37y - 2y^2) \bigg|_2^3$$

$$= \frac{1}{625} [(37)(3) - 18) - (74 - 8)]$$

$$= \frac{27}{625}$$

c.

$$f_X(x) = \int f(x,y) \, dy$$

$$= \frac{2}{625} \int_0^5 (20 - x - 2y) \, dy$$

$$= \frac{2}{625} [20y - xy - y^2]_0^5$$

$$= \frac{2}{625} (100 - 5x - 25)$$

$$= \frac{150 - 10x}{625}$$
\[ f_Y(y) = \int f(x, y) \, dx \]
\[ = \frac{2}{625} \int_0^5 (20 - x - 2y) \, dx \]
\[ = \frac{2}{625} [20x - \frac{x^2}{2} - 2xy]^5_0 \]
\[ = \frac{1}{625} [200 - 25 - 20y] \]
\[ = \frac{175 - 20y}{625} \]

d. The ethanol concentration and acidity are independent if and only if \( f(x, y) = f_X(x)f_Y(y) \). We see this is not the case, because \( f_X(x)f_Y(y) = \frac{150 - 10x + 175 - 20y}{625} \). Without computing the product we can see that this is not equal to \( f(x, y) = \frac{20 - x - 2y}{625} \) because it contains the cross-term 300xy. So \( X \) and \( Y \) are not independent.

e.

f.

g. \( f_{Y|X=3} = \frac{f(3,y)}{f_X(3)} = \frac{2(17-2y)}{60} \).

2. Question 2.6.10.

a.

\[
1 = A \int_0^L x(L - x) \, dx \\
1 = A \int_0^L Lx - x^2 \, dx \\
1 = A[Lx^2 \frac{2}{2} - \frac{x^3}{3}]_0^L \]
\[1 = A(\frac{L^3}{2} - \frac{L^3}{3}) \]
\[1 = A(\frac{L^3}{2} - \frac{L^3}{3}) \]
\[1 = A \frac{L^3}{6} \]
\[A = 6L^{-3} \]

b. Define a new random variable \( Y \) for the difference in length of the two pieces of the rod,
\[ Y = |L - 2X| \text{. Then,} \]
\[
F_Y(y) = P(Y \leq y) = P(-y \leq 2X - L \leq y) \\
= P((L - y)/2 \leq X \leq (L + y)/2) \\
= P(\frac{L - 2y}{2} \leq X) \\
= A \int_{(L+y)/2}^{(L-y)/2} x(L - x) \, dx \\
= Lx - x^2/2 \left( \frac{L+y}{2} \right)^2 - \left( \frac{L-y}{2} \right)^2 \\
= AL^2y/4 - Ay^3/12 = (3/2)y/L - y^3L^{-3}/2.
\]
\[ f_Y(y) = \frac{3}{2}/L - \frac{3}{2}y^2L^{-3}. \]

c.
\[
E(Y) = \int_0^L y\left[ (3/2)y/L - y^3L^{-3}/2 \right] \, dy \\
= \left. (1/2)y^2/L - y^4L^{-3}/8 \right|_0^L \\
= (1/2)L - (1/8)L = 3L/8.
\]

3. Question 3.1.6.

When choosing from among five answers, the probability of a correct answer on any question is \( .2 \). Then the random variable describing the number of correct answers is \( X \sim \text{Bin}(10, 1/5) \). The probability of passing is
\[
P(X \geq 7) = \binom{10}{7}(1/5)^7(4/5)^3 + \binom{10}{8}(1/5)^8(4/5)^2 \\
+ \binom{10}{9}(1/5)^9(4/5)^1 + \binom{10}{10}(1/5)^{10}(4/5)^0 \\
= 5^{-10}\left( \frac{10 \times 9 \times 8}{3 \times 2 \times 1} \right) \frac{4^3}{4^3} + \frac{10 \times 9}{2 \times 1} \frac{4^2}{4^2} + \frac{10}{1} \frac{4}{4} + 1 \\
= 5^{-10}(120 \times 64 + 45 \times 16 + 10 \times 4 + 1) \\
= 5^{-10} \times 8441 = 8.643 \times 10^{-4}.
\]

You could also have done this in R using `1-pbinom(6,10,.2)` to get the same answer.

When choosing from among two answers, the probability of a correct answer on any question
is $1/2$, and $X \sim \text{Bin}(10, 1/2)$. The probability of passing is
\[
P(X \geq 7) = \binom{10}{7} \left(\frac{1}{2}\right)^7 \left(\frac{1}{2}\right)^3 + \binom{10}{8} \left(\frac{1}{2}\right)^8 \left(\frac{1}{2}\right)^2 \\
+ \binom{10}{9} \left(\frac{1}{2}\right)^9 \left(\frac{1}{2}\right)^1 + \binom{10}{10} \left(\frac{1}{2}\right)^{10} \left(\frac{1}{2}\right)^0 \\
= 2^{-10} \left(\frac{10 \times 9 \times 8}{3 \times 2 \times 1} + \frac{10 \times 9}{2 \times 1} + \frac{10}{1} + 1\right) \\
= 2^{-10} (120 + 45 + 10 + 1) \\
= 2^{-10} \times 176 = .1719,
\]
or $1 - \text{pbinom}(6, 10, 1/2)$.

4. Question 3.2.6.

a. $X \sim \text{Geom}.37$; $E[X] = 1/p = 1/.37 = 2.70$.
b. $X \sim \text{NBin}(3, .37)$; $E[X] = r/p = 3/.37 = 8.11$.
c. Assume that, as in part b, 3 hits are needed to provide enough supplies. In this case, again $X \sim \text{NBin}(3, .37)$, and
\[
P(X \leq 10) = \binom{2}{2} (1 -.37)^0 .37^3 + \binom{3}{2} (1 -.37)^1 .37^3 \\
+ \binom{4}{2} (1 -.37)^2 .37^3 + \binom{5}{2} (1 -.37)^3 .37^3 \\
+ \binom{6}{2} (1 -.37)^4 .37^3 + \binom{7}{2} (1 -.37)^5 .37^3 \\
+ \binom{8}{2} (1 -.37)^6 .37^3 + \binom{9}{2} (1 -.37)^7 .37^3 \\
= .37^3 (1 + 3 \times .63^1 + 6 \times .63^2 + 10 \times .63^3 + \\
15 \times .63^4 + 21 \times .63^5 + 28 \times .63^6 + 36 \times .63^7) \\
= .7794.
\]
Alternatively, one might do $\text{npbinom}(7, 3, .37)$. Note that the first argument is the number of misses rather than the number of trials.

Alternatively, if $Y \sim \text{Binom}(10, .37)$, then $P(X \leq 10) = P(Y \geq 3) = 1 - P(Y \leq 2) = 1 - \binom{10}{0} .37^0 .63^{10} - \binom{10}{1} .37^1 .63^9 - \binom{10}{2} .37^2 .63^8 = .7794$, or $1 - \text{pbinom}(2, 10, .37)$.
d. $P(X = 10) = \binom{10}{2} (1 -.37)^7 .37^3 = .37^3 \times 36 \times .63^7 = 0.0718$. 

4
5. Question 3.3.8.

**Set up the problem like this:**

<table>
<thead>
<tr>
<th>Chocolate</th>
<th>Strawberry</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Child leaves</td>
<td>X</td>
<td>10</td>
</tr>
<tr>
<td>Child takes</td>
<td>9</td>
<td>6</td>
</tr>
</tbody>
</table>

We need \( P(5 \leq X \leq 7) \). The probability mass function is \( P(X = x) = \binom{7}{x}\binom{6}{10-x}/\binom{15}{10} \). Then \( P(5 \leq X \leq 7) = \binom{7}{5}\binom{6}{5} + \binom{7}{6}\binom{5}{5} + \binom{7}{7}\binom{4}{5}/\binom{15}{10} = (126 \times 6 + 84 \times 15 + 36 \times 20)/3003 = 0.9111 \). This could also have been done with \( \text{phyper}(7,10,5,9) - \text{phyper}(4,10,5,9) \).

6. Question 3.4.4.

Let \( X \) be the number of cracks. Then \( P(X = 0) = \exp(-2.4)(2.4)^0/0! = 0.0907 \). Also, \( P(X \geq 4) = 1 - P(X \leq 3) = \exp(-2.4)((2.4)^0/0! + (2.4)^1/1! + (2.4)^2/2! + (2.4)^3/3!) = 0.2212 \).

7. Question 3.5.2.

a. Consider the die outcomes as 6, with probably \( 1/6 \), 5, with probably \( 1/6 \), and anything else, with probably \( 2/3 \). The the probability of 3 6’s, 3 5’s, and 9 of anything else, is

\[
\begin{align*}
\text{Chocolate} & \quad 4^{9}/6^{15} = \frac{15 \times 14 \times 13 \times 12 \times 11 \times 10}{6 \times 6} \times 4^9/6^{15} = 5 \times 14 \times 13 \times 11 \times 10 \times 4^9/6^{15} = 0.0558.
\end{align*}
\]

b. Consider the die outcomes as 6, with probably \( 1/6 \), 5, with probably \( 1/6 \), 4, with probably \( 1/6 \), and anything else, with probably \( 1/2 \). The the probability of 3 6’s, 3 5’s, 4 4’s, and 9 of anything else, is

\[
\begin{align*}
\text{Chocolate} & \quad \frac{15}{2} \times 3^5/6^{15} = \frac{15 \times 14 \times 13 \times 12 \times 11 \times 10 \times 9 \times 8 \times 7}{6 \times 6 \times 24} \times 3^5/6^{15} = 5 \times 7 \times 13 \times 11 \times 5 \times 9 \times 8 \times 73^5/6^{15} = 0.0065.
\end{align*}
\]

c. Consider the die outcomes as 6, with probably \( 1/6 \), and anything else, with probably \( 5/6 \). The probability of two 6’s is

\[
\begin{align*}
\text{Chocolate} & \quad \frac{15}{2} \times 5^{13}/6^{15} = 105 \times 5^{13}/6^{15} = 0.2726.
\end{align*}
\]

8. Question 3.7.12.

a. Let \( p \) be the probability of heads. Then in order for the two probabilities to be the same, \( \binom{1}{3}p^3(1-p)^4 = \binom{3}{7}p^4(1-p)^3 \). Since \( \binom{1}{3} = \binom{3}{7} \), this implies that \( p^3(1-p)^4 = p^4(1-p)^3 \), or \( 1-p = p \), or \( p = \frac{1}{2} \). But \( p \neq \frac{1}{2} \), and so the statement is true.

b. The probability, from the geometric distribution, is \( (5/6)^7(1/6)^1 = \frac{78125}{1679616} \). True.

c. The probability, from the geometric distribution, is \( \binom{6}{3}(1/2)^3 = \frac{20}{128} = \frac{5}{32} \). True.

d. The variance of the number of heads is \( 16 \times \frac{1}{2} \times \frac{1}{2} = 4 \), and the variance of the proportion is \( 4/16^2 = 1/64 \), making the standard deviation \( 1/8 \). True.