Lecture 13

H. Relationship among distributions:

<table>
<thead>
<tr>
<th>Geometric (Number of Bernoulli trials until first success)</th>
<th>Make $r = 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Negative Binomial (Number of Bernoulli trials until $r$ successes)</td>
<td>Count trials until $r$ successes</td>
</tr>
<tr>
<td>Bernoulli trial (Random event that takes on two values, one with prob. $p$)</td>
<td>Count successes from $n$ Bernoulli trials</td>
</tr>
<tr>
<td>Binomial (Number of successes from $n$ Bernoulli trials)</td>
<td>Change $p$ to reflect draws w/o replacement, or condition on sum of two ind. binomials</td>
</tr>
<tr>
<td>Let $p \to 0$, $n \to \infty$ so that $pn = \lambda$</td>
<td>Hypergeometric</td>
</tr>
</tbody>
</table>

IV. Important continuous distributions

: 4.1-4.3
A. Uniform distribution \( X \sim U(a, b) \).

1. pdf \( f_X(x) = \begin{cases} 
1/(b-a) & \text{if } x \in [a, b] \\
0 & \text{otherwise.} 
\end{cases} \)

2. cdf \( F_X(x) = \begin{cases} 
1/(b-a) & \text{if } x \in [a, b] \\
1 & \text{if } x > b. 
\end{cases} \)

3. Expectation \( E[X] = \int_a^b x/(b-a) \, dx = (b^2/2 - a^2/2)/(b-a) = (a + b)/2. \)
   
a. We could have seen this through symmetry.
   
b. Median is the same.

4. Variance: \( E[X^2] = \int_a^b x^2/(b-a) \, dx = (b^3/3 - a^3/3)/(b-a) = (a^2 + ab + b^2)/3, \) and so \( \text{Var}[X] = (a^2 + ab + b^2)/3 - (a^2 + 2ab + b^2)/4 = (a^2 - 2ab + b^2)/12 = (a - b)^2/12. \)

B. Exponential distribution \( X \sim \text{Expon}(\lambda) \).

1. We’ve seen this before

2. pdf \( f_X(x) = \lambda \exp(-\lambda x) \)

3. cdf \( F_X(x) = \int_0^x \lambda \exp(-\lambda y) \, dy = \int_0^x \lambda \exp(-z) \, dz = -\exp(-z)|_0^x \lambda = 1 - \exp(-x \lambda). \)

4. Expectation \( E[X] = \int_0^\infty y \lambda \exp(-\lambda y) \, dy = \lambda^{-1} \int_0^\infty z \exp(-z) \, dz = \lambda^{-1}(-\exp(-z)|_0^\infty - \int_0^\infty (-\exp(-z)) \, dz = \lambda^{-1}(0 + 1) = \lambda^{-1}. \)
5. Median $\mu$ satisfies $F_X(\mu) = 1/2$, which implies 
\[ \exp(-\lambda \mu) = 1/2, \text{ or } \mu = \log(2)/\lambda. \]

6. Variance: 
\[ E[X^2] = \int_0^\infty y^2 \lambda \exp(-\lambda y) \, dy = \lambda^{-2} \int_0^\infty z^2 \exp(-z) \, dz = \lambda^{-2} (-z^2 \exp(-z)|_0^\infty - 2 \int_0^\infty (-z \exp(-z)) \, dz = 2\lambda^{-2}, \text{ and } \Var[X] = 2\lambda^{-2} - (\lambda^{-1})^2 = \lambda^{-2}. \]

7. Memoryless property: For $x, y > 0$, 
\[ P(X \geq y + x | X \geq x) = \exp(-\lambda(y + x))/\exp(-\lambda x) = \exp(-\lambda y). \]

   a. Probability of lasting into the future, conditional on being around now, doesn't depend on how old it is now, or "new as good as used".

   b. Distribution of the sum of two independent copies:
\[ X \sim \text{Expon}(\lambda), Y \sim \text{Expon}(\lambda), X \perp Y, Z = X + Y. \]

Then 
\[ P(Z \leq z) = \int_0^z \int_0^{z-y} \lambda^2 \exp(-\lambda(x+y)) \, dx \, dy = \int_0^z \exp(-\lambda y) \lambda \int_0^{z-y} \lambda \exp(-\lambda x) \, dx \, dy = \int_0^z \exp(-\lambda y) \lambda [1 - \exp(-\lambda(z-y))] \, dy = \int_0^z \lambda [\exp(-\lambda y) - \exp(-\lambda z)] \, dy = \int_0^z \lambda \exp(-\lambda y) \, dy - \int_0^z \lambda \exp(-\lambda z) \, dy = 1 - \exp(-\lambda z) - z \lambda \exp(-\lambda z) \quad f_Z(z) = z \lambda^2 \exp(-\lambda z) \]

8. Distributions like this are called $\Gamma$ distributions.
a. $\Gamma(k, \lambda)$ density is $\exp(-\lambda z)z^{k-1}\lambda^k/A$.

b. Constant $A$ is whatever it takes to make this integrate to one.

c. $A = \int_0^\infty \exp(-z)z^k$, called $\Gamma(k)$.

d. We see from above that $\Gamma(1) = \Gamma(2) = 1$.

e. You can show easily that $\Gamma(k) = (k - 1)!$.

f. You can show, with some more trouble, that if $X \sim \Gamma(k, \lambda)$, $Y \sim \Gamma(j, \lambda)$, $X \perp Y$, then $X + Y \sim \Gamma(j + k, \lambda)$.

g. Note that $\text{Expon}(\lambda) = \Gamma(1, \lambda)$.

9. If $X \sim \Gamma(k, \lambda)$, then

$$E[X] = \int_0^\infty xx^{k-1}\lambda^k \exp(\lambda x)/\Gamma(k) \, dx$$

$$= \lambda^{-1} \int_0^\infty x^k \lambda^{k+1} \exp(\lambda x)/\Gamma(k) \, dx$$

$$= \lambda^{-1} \Gamma(k + 1)/\Gamma(k) = k\lambda^{-1}.$$