d. Relation to p.d.f. for continuous distributions:
   i. \( F_X(x) = \int_{-\infty}^{x} f_X(y) \, dy \).
   ii. If \( f_X \) is continuous at \( x_1 \), and if \( x_2 \) is close to \( x_1 \), and \( x_2 > x_1 \), then
       \( F_X(x_2) - F_X(x_1) = P(x_1 < X \leq x_2) = \int_{x_1}^{x_2} f_X(y) \, dy \approx (x_2 - x_1)f_X(x_1) \).
   iii. Hence \( (F_X(x_2) - F_X(x_1))/(x_2 - x_1) \approx f_X(x_1) \).
   iv. Hence \( dF_X(x_1)/dx_1 = f_X(x_1) \): Fundamental theorem of calculus.
   v. Example: \( f_X(x) = 1 \) for \( x \in (0, 1) \), and equal 0 elsewhere.
      • Then
        \[ F_X(x) = \begin{cases} 
        0 & \text{if } x \leq 0 \\
        x & \text{if } x \in (0, 1) \\
        1 & \text{if } x \geq 1
        \end{cases} \]
      • Then
        \[ F_X(x) = \begin{cases} 
        0 & \text{if } x < 0 \\
        1 & \text{if } x \in (0, 1) \\
        0 & \text{if } x > 1 \\
        \text{undefined} & \text{if } x \in \{0, 1\}
        \end{cases} \]

M. Describing Distributions:
1. Typical Values
   a. The expectation, mean, or average value.
      i. Definition:
         • for continuous dists as \( \int x f_X(x) \, dx \)
         • Example: Exponential distribution with \( f_X(x) = \exp(-x) \) for \( x \in [0, \infty) \).
   b. Integration by parts shows expectation is 1.
   c. for discrete dists as \( \sum x p_X(x) \).
      • Example: Count from one die:
        \[ E[=]1 \times \frac{1}{6} + 2 \times \frac{1}{6} + 3 \times \frac{1}{6} + 4 \times \frac{1}{6} + 5 \times \frac{1}{6} + 6 \times \frac{1}{6} = 21/6 = 3.5 \] Note that this is not a potential data value.
      • Example: Count from two dice:
        \[
        \begin{array}{ccc}
        x & p_X(x) & xp_X(x) \\
        2 & 1/36 & 1/18 \\
        3 & 2/36 & 3/18 \\
        4 & 3/36 & 6/18 \\
        5 & 4/36 & 10/18 \\
        6 & 5/36 & 15/18 \\
        7 & 6/36 & 21/18 \\
        8 & 5/36 & 20/18 \\
        9 & 4/36 & 18/18 \\
        10 & 3/36 & 15/18 \\
        11 & 2/36 & 11/18 \\
        12 & 1/36 & 6/18 \\
        \end{array}
        \]
        Sum of last column is \( 128/18 = 7 \).
        • Note that expectation is point of symmetry, if distribution is symmetric.
        • Note that expectation of sum is sum of expectations: always holds.

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