N. Common calculation tool:

2. \( \text{call} E[X^k] \text{ the moment of order } k \text{ of } X \)

O. Example: \( f_X(x) = \exp(-x) \) for \( x \in [0, \infty) \).

1. As before, \( E[X] = 1 \).
2. \( E[X^2] = \int_0^\infty x^2 \exp(-x) \, dx = -x^2 \exp(-x)|_0^\infty - \left( - \int_0^\infty 2x \exp(-x) \, dx \right) = 0 - 0 + 2 \int_0^\infty 2x \exp(-x) \, dx = 2E[X] = 2. \)
3. \( \text{Var}[X] = 2 - 1 = 1. \)
4. Standard deviation is 1.
5. Alternative measure of spread:
   a. For \( q \in (0, 1) \), define the \( q \) quantile of a distribution to be that value \( x \) such that \( P(X \leq x) \geq q \)
   b. So the median is the \( \frac{1}{2} \) quantile
   c. Call the \( \frac{1}{4} \) and \( \frac{3}{4} \) quantiles the upper and lower quartiles.
   d. Distance between these represents a typical measure of how spread out the values of the random variable are
   e. Call the distance the interquartile range: 2.5

P. For multiple random variables, consider the probabilities of events defined in terms of combinations of these variables.

1. Ex., red die and white die; what is the probability that roll on both is greater than 3?
2. Ex., 100 people are surveyed and asked their preference for one of aspirin, ibuprofen

\[ P(X + Y \leq z) = \int_0^z \int_0^{z-x} \exp(-x) \exp(-y) \, dy \, dx \]
\[ = \int_0^z \exp(-x)(1 - \exp(-z)) \, dx \]
\[ = \int_0^z (\exp(-x) - \exp(-z)) \, dx \]
\[ = 1 - \exp(-z) - z \exp(-z) \]

i. \( f_Z(z) = F'_Z(z) = \exp(-z) + z \exp(-z) - \exp(-z) = z \exp(-z). \)

6. Can recover pdfs, pmfs, and cdfs for one variable from joint distributions: called marginal quantities.

a. If \( X \) and \( Y \) are discrete, then \( P(X = x) = P(X = x \text{ and } Y = \text{ anything}) = \sum_{y \in Y} p_{X,Y}(x, y) \)
   and \( P(X \leq x) = P(X \leq x \text{ and } Y = \text{ anything}) = \sum_{z \in Z, z \leq x} \sum_{y \in Y} p_{X,Y}(z, y) \)

b. If \( X \) and \( Y \) are continuous, then \( F_Z(x)P(X \leq x) = P(X \leq x \text{ and } Y = \text{ anything}) = \int_{-\infty}^x \int_{-\infty}^{z \leq z} f_{X,Y}(z, y) \)
   and \( f_Z(x) = F'_Z(x) = \int_{-\infty}^x f_{X,Y}(x, y) \)

7. Measure of Association

a. Make score for various combinations of potential values for the pair of variables.
   b. Score will be + if it provides evidence that variables are moving together in a positive direction
   i. i.e., high \( X \) matched with high \( Y \) and low \( X \) matched with low \( Y \).
   c. Score will be - if it provides evidence that variables are moving together in a negative direction

a. Asked the same question twice.

b. \( X \) and \( Y \) are numbers preferring ibuprofen first and second time resp.

c. If they really are the same, what is \( P(X \geq 60, Y \geq 60) \)?

d. Hard, because a person is likely to answer the same way both times.

e. Need mass function for the combination.

f. \( X \) and \( Y \) are called independent if probabilities of sets they generate are independent
   i. if and only if pmfs (or densities) multiply

3. Consider random variables \( X, Y, \ldots \) with supports \( X, Y, \ldots \).

4. For discrete random variables,

a. let \( p_{X,Y,\ldots}(x,y,\ldots) \) be the joint pmf
   \[ P(X = x, Y = y, \ldots) \text{ for } (x, y, \ldots) \in X \times Y \times \cdots \]

b. \( \times \) notation indicates the set of vectors in which each component comes from respective smaller space.

5. For continuous random variables,

a. have joint density \( f_{X,Y,\ldots}(x,y,\ldots) \), and

b. probabilities are generated by bi variate integrals.

c. Ex., \( X \) and \( Y \) are independent exponential random variables.

i. i.e., low \( X \) matched with high \( Y \) and high \( X \) matched with low \( Y \).

\[ \text{d. Result is combination of score values over all possible values for pair.} \]

i. + for positive association, - for negative association, \( 0 \) for no association on average

e. \( \text{Cov}[X,Y] = E[(X-E[X])(Y-E[Y])]. \)

\[ \text{f. Called covariance.} \]