E. Multinomial Distribution
1. Suppose $n$ items are divided into $k \geq 3$ classes, and results of experiment are counts $X_1, \ldots, X_k$ for each class.
2. Probabilities for the various groups are $p_1, \ldots, p_k$.
3. Probabilities for a string of outcomes is the product of $p$’s with subscript given by output.
4. Probability of string depends only on the total number for each kind, and not on the order that the results come in.
5. Number of such strings is $\frac{n!}{x_1! \cdots x_k!} p_1^{x_1} \cdots p_k^{x_k}$.
6. $P(X_1 = x_1, \ldots, X_k = x_k) = \frac{n!}{x_1! \cdots x_k!} p_1^{x_1} \cdots p_k^{x_k}$.
7. Individually, $X_j \sim \text{Bin}(n, p_j)$, but they aren’t independent.

F. Hyper geometric Distribution
1. Definitions:
   a. $N$ tickets in box
   b. $r$ of them are red
   c. Draw $n$ of them.
   d. How many of drawn tickets are red?

<table>
<thead>
<tr>
<th>Drawn</th>
<th>Blue</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x$</td>
<td>$n-x$</td>
<td>$n$</td>
</tr>
<tr>
<td>$r-x$</td>
<td>$N-r-n+x$</td>
<td>$N-n$</td>
</tr>
<tr>
<td>$r$</td>
<td>$N-r$</td>
<td>$N$</td>
</tr>
</tbody>
</table>
2. PMF:
   a. If $n = 1$ then $X$ is $B(n, r/N)$.
   b. For larger $n$ this is no longer true, since if the first ticket is red, the second is less likely to be red, and vice versa.
   c. The probability of getting $x$ red tickets first, and then $n-x$ blue tickets, is
      $$P(X = x) = \frac{\binom{r}{x} \binom{N-r}{n-x}}{\binom{N}{n}}\text{ for } r \leq x \leq N-r.$$  
   d. Any other sequence of red and blue giving $x$ red has same probability.
   e. There are $\binom{n}{x}$ such sequences.
   f. Total probability is $\binom{n}{r} \binom{N-r}{n-r}$.
   g. As long as $\max(0, n+r-N) \leq x \leq \min(n, r)$.
   h. Distribution remains the same if we interchange role of $r$ and $n$, or rows and columns of table.
3. If $X \sim \text{Bin}(r, p), Y \sim \text{Bin}(N-r, p), X \perp Y$, then
   a. $X + Y \sim \text{Bin}(N, p)$

\[
P(X = x|X + Y = n) = \frac{P(X = x, Y = n-x)}{P(X + Y = n)}
\]

\[
= \frac{\binom{r}{x} \binom{N-r}{n-x} p^x (1-p)^{n-x}}{\binom{N}{n} p^n (1-p)^{N-n}}
\]

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