1. A random variable $X$ has the density $\exp(-|x|)/2$ for $x \in (-\infty, \infty)$. Calculate the moment generating function $m_X(t)$ for $X$. Include information on which values of $t$ make $m_X(t)$ finite.

$$m_X(t) = \int_{-\infty}^{\infty} \exp(tx) \exp(-|x|)/2 \, dx$$

$$= \int_{-\infty}^{0} \exp(tx) \exp(x)/2 \, dx + \int_{0}^{\infty} \exp(tx) \exp(-x)/2 \, dx$$

$$= \int_{0}^{\infty} \exp(-tx) \exp(-x)/2 \, dx + \int_{0}^{\infty} \exp(tx) \exp(-x)/2 \, dx$$

The first integral is finite if $1 + t > 0$, and the second integral is finite if $1 - t > 0$. Both of these inequalities are true if $|t| < 1$. For such values of $t$, change variables to obtain

$$m_X(t) = \int_{0}^{\infty} \frac{\exp(-y)}{2(1 + t)} \, dy + \int_{0}^{\infty} \frac{\exp(-y)}{2(1 - t)} \, dy$$

$$= (1 + t)^{-1}/2 + (1 - t)^{-1}/2.$$ 

This is finite as long as $1 + t$ and $1 - t$ are both positive, or if $|t| < 1$.

2. Two probabilists consider a random variable taking a value between zero and one, with probability spread evenly over this region. Probabilist A uses the density $a_X(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$. Probabilist B uses the density $b_X(x) = \begin{cases} 1 & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$.

Which of these is the right density, or does the choice not matter? The two densities are entirely equivalent, because integrals with the integrand changed at a countable number of points (two here, at 0 and 1) are the same.

3. Suppose that a random variable $X$ has probability function $f_X(x) = A/x$ for $x \in (1, 2)$.

a. What is the value of the constant $A$?

Note that $1 = \int_{1}^{2} f_X(x) \, dx = \int_{1}^{2} A/x \, dx = A \int_{1}^{2} 1/x \, dx = A(\log(2) - \log(1)) = A \log(2)$. Hence $A = 1/\log(2) = 1.443$.

b. Calculate $E(X)$.

$E(X) = \int_{1}^{2} x f_X(x) \, dx = \int_{1}^{2} A x \, dx = A \int_{1}^{2} x \, dx = 1/\log(2) = 1.443$.

c. Calculate the median of $X$.

The median $m$ satisfies $1/2 = \int_{1}^{m} f_X(x) \, dx = (\log(2))^{-1} \int_{1}^{m} 1/x \, dx = (\log(2))^{-1} \log(m)$.

Then $\log(m) = (1/2) \log(2) = \log(\sqrt{2})$, and $m = \sqrt{2} = 1.414$.

d. Calculate $V(X)$.

$E(X^2) = \int_{1}^{2} x^2 f_X(x) \, dx = \int_{1}^{2} A x \, dx = A \int_{1}^{2} x \, dx = A x^2/2|_1^2 = A(2 - 1/2) = 3/(2 \log(2))$, and so $V(X) = E(X^2) - E(X)^2 = 3/(2 \log(2)) - (1/\log(2))^2 = 0.08267358$.

e. Calculate $P(X \leq 3/2)$.
\[ P(X \leq 3/2) = \int_{1}^{3/2} f_X(x) \, dx = \log(3/2)/\log(2) = \log(3)/\log(2) = 10.5850. \]

4. Suppose that a random variable \( X \) has expectation 0 and variance 1. Give a bound to \( P(|X| \geq 2) \).

*Use the Tchebysheff inequality to show* \( P(|X| \geq 2) \leq \frac{V(X)}{2^2} = 1/4 \).

5. Suppose that a random variable \( X \) has probability function

\[ f_X(x) = \exp(-x^2/2)/\sqrt{2\pi}, \]

and let \( Y = \exp(X) \) (that is, \( Y = e^X \) for \( e \) the base of the natural log). Give the density for \( Y \). Be sure to specify the set of \( Y \) values on which its density is positive.

*Since \( r(x) = \exp(x) \), then \( r^{-1}(y) = \log(y) \), for \( \log \) the natural log with base \( e \). Hence*

\[ f_Y(y) = f_X(r^{-1}(y)) \frac{d}{dy} r^{-1}(y) = \exp(-\log(y)^2/2) \frac{1}{\sqrt{2\pi} y} \text{ for } y > 0, \text{ and } 0 \text{ otherwise.} \]