1. Question 2.5.8 (Hayter), page 127. The random variable \( X \) measures the concentration of ethanol in a chemical solution, and the random variable \( Y \) measures the acidity of the solution. They have a joint probability density function
\[
f(x, y) = A(20 - x - 2y)
\]
for \( 0 \leq x \leq 5 \) and \( 0 \leq y \leq 5 \), and \( f(x, y) = 0 \) elsewhere.

a. What is the value of \( A \)?

Set \( \int f(x, y) \, dx \, dy = 1 \). Then, \( \int_0^5 \int_0^5 A(20 - x - 2y) \, dx \, dy = 1 \). So to determine \( A \), set the integral of the density to 1:
\[
1 = \int_0^5 \int_0^5 A(20 - x - 2y) \, dx \, dy.
\]
Performing the inner integral with respect to \( x \) gives
\[
1 = \int_0^5 A(20x - \frac{x^2}{2} - 2xy) \Big|_0^5 \, dy = \int_0^5 A(100 - \frac{25}{2}y - 10y) \, dy.
\]
Integrating with respect to \( y \) gives
\[
1 = A(100y - \frac{25}{2}y^2 - 5y^2) \Big|_0^5 = A(500 - \frac{125}{2} - 125) = A \frac{625}{2}.
\]
Setting the integral to 1, \( A = \frac{2}{625} \).

b. What is \( P(1 \leq X \leq 2, 2 \leq Y \leq 3) \)?

The probability is given by the integral
\[
P(1 \leq X \leq 2, 2 \leq Y \leq 3) = \int_2^3 \int_1^2 \frac{2}{625}(20 - x - 2y) \, dx \, dy.
\]
Integrate with respect to \( x \) to give \( P(1 \leq X \leq 2, 2 \leq Y \leq 3) \) as
\[
\int_3^2 \frac{2}{625} \left(20x - \frac{x^2}{2} - 2xy\right) \Big|_1^2 \, dy = \frac{1}{625} \int_3^2 (37 - 4y) \, dy.
\]
Integrate with respect to \( y \) to give \( P(1 \leq X \leq 2, 2 \leq Y \leq 3) \) as
\[
\frac{1}{625}(37y - 2y^2) \Big|_2^3 = \frac{1}{625}[((37)(3) - 18) - (74 - 8)] = \frac{27}{625}.
\]

c. If the ethanol concentration is 3, what is the conditional probability density function of the acidity?

\[
f_{Y|X=3}(y) = \frac{f(3,y)}{f_X(3)} = \frac{2(17-2y)}{60}
\]

\[.\]

d. What is the covariance between ethanol concentration and the acidity?

We previously showed that \( A = \frac{2}{625} \), and that \( E(X) = \frac{7}{3} \) and \( E(Y) = \frac{13}{6} \). Then \( E(XY) = \int xyf(x,y) \, dx \, dy = \int_0^5 \int_0^5 Axy(20 - x - 2y) \, dx \, dy \). Integrating
with respect to \( x \), \( E(XY) = \int_0^5 y A \left( 10x^2 - \frac{x^3}{3} - x^2y \right) dy \). This simplifies to \( E(XY) = \int_0^5 yA \left( \frac{625}{3} - 25y \right) dy \). Integrating with respect to \( y \),
\[
E(XY) = A \left( \frac{625}{6} y^2 - \frac{25}{3} y^3 \right) \bigg|_0^5 = \frac{2}{625} \left( \frac{625 \times 25}{6} - \frac{5 \times 625}{3} \right) = 5.
\]
Then \( \text{Cov}(X,Y) = E(XY) - E(X)E(Y) = 5 - \frac{91}{18} = -\frac{1}{18} \).

2. Question 2.5.8 (Hayter), page 127. The random variable \( X \) measures the concentration of ethanol in a chemical solution, and the random variable \( Y \) measures the acidity of the solution. They have a joint probability density function
\[
f(x, y) = A(20 - x - 2y)
\]
for \( 0 \leq x \leq 5 \) and \( 0 \leq y \leq 5 \), and \( f(x, y) = 0 \) elsewhere.

a. Construct the marginal probability density functions \( f_X(x) \) and \( f_Y(y) \).

Construct the marginal density by integrating the joint density over the variable not of interest:
\[
f_X(x) = \int_0^5 f(x, y) dy = \int_0^5 \frac{2(20 - x - 2y)}{625} dy = \frac{2[20y - xy - y^2]}{625} \bigg|_0^5 = \frac{6}{25} - \frac{2x}{125}
\]
for \( x \in (0, 5) \). Furthermore, \( f_X(x) = 0 \) for \( x \notin (0, 5) \). Also,
\[
f_Y(y) = \int f(x, y) dx = \int_0^5 \frac{2(20 - x - 2y)}{625} dx = \frac{2[20x - \frac{x^2}{2} - 2xy]}{625} \bigg|_0^5 = \frac{7}{25} - \frac{4y}{125}
\]
for \( y \in (0, 5) \). Furthermore, \( f_Y(y) = 0 \) for \( y \notin (0, 5) \).

b. Are ethanol concentration and acidity independent?

The ethanol concentration and acidity are independent if and only if \( f(x, y) = f_X(x)f_Y(y) \).
We see this is not the case, because \( f_X(x)f_Y(y) = \frac{150 - 10x}{625} \frac{175 - 20y}{625} \) Without computing the product we can see that this is not equal to \( f(x, y) = \frac{20 - x - 2y}{625} \) because it contains the cross-term 300xy. So \( X \) and \( Y \) are not independent.

c. What are the expectation and the variance of the ethanol concentration?

The expectation of \( X \) is \( \int_0^5 x(6/25 - 2x/125) dx = (6x^2/50 - 2x^3/375) \bigg|_0^5 = 3 - 2/3 = 7/3 \),
and the second moment is \( \int_0^5 x^2(6/25 - 2x/125) dx = (2x^3/25 - x^4/250) \bigg|_0^5 = 10 - 5/2 = 15/2 \). Hence the variance is \( 15/2 - 49/9 = (135 - 98)/18 = 37/18 \).

d. What are the expectation and the variance of the acidity?

\[
E(Y) = \int_0^5 y(7/25 - 4y/125) dy = \frac{7y^2}{50} - \frac{4y^3}{375} \bigg|_0^5 = \frac{7}{2} - \frac{4}{3} = \frac{13}{6}
\]
and \( E(Y^2) = \int_0^5 y^2(7/25 - 4y/125) dy = \frac{7y^3}{75} - \frac{y^4}{125} \bigg|_0^5 = \frac{35}{3} - 5 = \frac{20}{3} \). Hence
\[
V(Y) = \frac{20}{3} - 169/36 = \frac{71}{36}.
\]
e. If the ethanol concentration is 3, what is the conditional probability density function of the acidity?
Similarly, the conditional density for the music score given the math score is
\[ f_{Y|X=3}(y) = \frac{f(3,y)}{f_X(3)} = \frac{2(17-2y)}{60} . \]

f. What is the covariance between ethanol concentration and the acidity?

We previously showed that \( A = 2/625 \), and that \( E(X) = 7/3 \) and \( E(Y) = 13/6 \). Then \( E(XY) = \int xyf(x,y)dx \ dy = \int_0^5 f_0^5 Axy(20 - x - 2y)dy \ dx \). Integrating with respect to \( x \), \( E(XY) = \int_0^5 y A \left( 10x^2 - \frac{25}{3} - x^2y \right) \bigg|_0^5 dy \). This simplifies to
\[ E(XY) = \int_0^5 y A \left( \frac{625}{6} - 25y \right) dy . \] Integrating with respect to \( y \),
\[ E(XY) = A \left( \frac{625}{6} y^2 - \frac{25}{3} y^3 \right) \bigg|_0^5 = \frac{2}{625} \left( \frac{625 \times 25}{6} - \frac{5 \times 625}{3} \right) = 5. \]

Then \( Cov(X,Y) = E(XY) - E(X)E(Y) = 5 - \frac{91}{18} = -\frac{1}{18} \).

3. Question 7 (DeGroot), page 140. Suppose that a person’s score \( X \) on a mathematics aptitude test is a number between 0 and 1, and that his score \( Y \) on a music aptitude test is a also number between 0 and 1. Suppose further that in the population of all college students in the United States, the scores \( X \) and \( Y \) are distributed according to the following joint probability density function
\[ f(x,y) = \begin{cases} \frac{2}{5}(2x + 3y) & \text{for } 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 1, \\ 0 & \text{otherwise.} \end{cases} \]

a. What proportion of college students obtain a score greater than 0.8 on the mathematics test?

The marginal density for math scores is \( \int_0^1 \frac{2}{5}(2x + 3y) \ dy = \frac{3}{5} + \frac{4x}{5} \). Integrating from .8 to 1 gives 0.264.

b. If a student’s score on the music test is 0.3, what is the probability that his score on the mathematics test will be greater than 0.8?

The marginal density for music scores is \( f_Y(y) = \int_X f_{X,Y}(x,y) \ dx = \int_0^1 \frac{2}{5}(2x + 3y) \ dx = \frac{2}{5} + \frac{6y}{5} \). Hence the conditional density for the math score given the music score is
\[ f_{X|Y}(x|y) = f_{X,Y}(x,y)/f_Y(y) = \frac{\frac{2}{5}(2x + 3y)}{\frac{2}{5} + \frac{6y}{5}} = (2x + 3y)/(1 + 3y) \); setting \( y = .3 \) gives the density \( f_{X|Y}(x,.3) = (2x + 0.9)/1.9 \). and integrating from .8 to 1 gives \( \int_{.8}^{1} (2x + 0.9)/1.9 \ dx = 0.284. \)

c. If a student’s score on the mathematics test is 0.3, what is the probability that his score on the music test will be greater than 0.8?

Similarly, the conditional density for the music score given the math score is \( \frac{2}{5}(2x + 3y)/(\frac{2}{5} + \frac{4x}{5}) = 2(2x + 3y)/(3 + 4x) \). Setting \( x = .3 \) gives 0.47619(0.6 + 3y).

Integrating from .8 to 1 gives 0.314.
4. Question 2.5.4 (Hayter), page 126. A fair coin is tossed four times, and the random variable $X$ is the number of heads in the first three tosses and the random variable $Y$ is the number of heads in the last three tosses.

a. What is the marginal probability functions of $X$ and $Y$?

As above, the joint distribution, with marginal sums included, is

<table>
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<tr>
<th>Y</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
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<td>1/16</td>
<td>0</td>
<td>0</td>
<td>1/8</td>
</tr>
<tr>
<td>1</td>
<td>1/16</td>
<td>3/16</td>
<td>2/16</td>
<td>0</td>
<td>3/8</td>
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<td>2</td>
<td>0</td>
<td>2/16</td>
<td>3/16</td>
<td>1/16</td>
<td>3/8</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>0</td>
<td>1/16</td>
<td>1/16</td>
<td>1/8</td>
</tr>
<tr>
<td>Total</td>
<td>1/8</td>
<td>3/8</td>
<td>3/8</td>
<td>1/8</td>
<td></td>
</tr>
</tbody>
</table>

Both have the same probability function. It is

$$P(X = x) = \begin{cases} 1/8 & \text{if } x = 0 \\ 3/8 & \text{if } x = 1 \\ 3/8 & \text{if } x = 2 \\ 1/8 & \text{if } x = 3 \end{cases}$$

This is the binomial probability function with 3 trials and .5 success probability.

b. Are the random variables $X$ and $Y$ independent?

No. Check, for ex, whether $1/16 = P(X = 0, Y = 0) = P(X = 0) P(Y = 0) = (1/8)^2$. This does not hold.

c. What are the expectations and variances of the random variables $X$ and $Y$?

The expectations arnd variances are exactly those for the binomial random variable, 1.5 and $.5 \times .5 = .25$ respectively.

d. If there is one head in the last three tosses, what is the conditional probability mass function for $X$? What is the conditional expectation and variance of $X$?

The column of the table of joint probabilities corresponding to $Y = 3$ is given in the second column below. The conditional distribution of interest is the third column.

| $x$ | $P(X = x, Y = 1)$ | $P(X = x | Y = 1)$ |
|-----|-------------------|-------------------|
| 0   | 1/16              | $(1/16)/(3/8) = 1/6$ |
| 1   | 3/16              | $(3/16)/(3/8) = 1/2$ |
| 2   | 2/16              | $(2/16)/(3/8) = 1/3$ |
| 3   | 0                 | 0                 |

The expectation of this distribution is $1/2 + 2 \times 1/3 = 7/6$. The variance is $1 \times 1/6 + (2/3)^2 \times 1/2 + (5/6)^2 \times 1/3 = 0.62$. 


5. Question 2.5.8 (Hayter), page 127. The random variable $X$ measures the concentration of ethanol in a chemical solution, and the random variable $Y$ measures the acidity of the solution. They have a joint probability density function

$$ f(x, y) = A(20 - x - 2y) $$

for $0 \leq x \leq 5$ and $0 \leq y \leq 5$, and $f(x, y) = 0$ elsewhere.

a. What is the value of $A$?

Set $\int f(x, y) \, dx \, dy = 1$. Then, $\int_{0}^{5} \int_{0}^{5} A(20 - x - 2y) \, dx \, dy = 1$. So To determine $A$, set the integral of the density to 1:

$$ 1 = \int_{0}^{5} \int_{0}^{5} A(20 - x - 2y) \, dx \, dy. $$

Performing the inner integral with respect to $x$ gives

$$ 1 = \int_{0}^{5} A(20x - \frac{x^2}{2} - 2xy) \bigg|_{0}^{5} \, dy = \int_{0}^{5} A(100 - \frac{25}{2} - 10y) \, dy. $$

Integrating with respect to $y$ gives

$$ 1 = A(100y - \frac{25}{2}y - 5y^2) \bigg|_{0}^{5} = A(500 - \frac{125}{2} - 125) = A \frac{625}{2}. $$

Setting the integral to 1, $A = \frac{2}{625}$.

b. What is $P(1 \leq X \leq 2, 2 \leq Y \leq 3)$?

The probability is given by the integral

$$ P(1 \leq X \leq 2, 2 \leq Y \leq 3) = \int_{1}^{2} \int_{2}^{3} \frac{2}{625} (20 - x - 2y) \, dx \, dy. $$

Integrate with respect to $x$ to give $P(1 \leq X \leq 2, 2 \leq Y \leq 3)$ as

$$ \int_{2}^{3} \frac{2}{625} (20x - \frac{x^2}{2} - 2xy) \bigg|_{1}^{2} \, dy = \frac{1}{625} \int_{2}^{3} (37 - 4y) \, dy. $$

Integrate with respect to $y$ to give $P(1 \leq X \leq 2, 2 \leq Y \leq 3)$ as

$$ \frac{1}{625} (37y - 2y^3) \bigg|_{2}^{3} = \frac{1}{625} ((37)(3) - 18) - (74 - 8)] = \frac{27}{625}. $$

c. Construct the marginal probability density functions $f_X(x)$ and $f_Y(y)$.

Construct the marginal density by integrating the joint density over the variable not of interest:

$$ f_X(x) = \int_{0}^{5} f(x, y) \, dy = \int_{0}^{5} \frac{2(20 - x - 2y)}{625} \, dy = \frac{2[20y - xy - y^2|_{0}^{5}]}{625} = \frac{6}{25} - \frac{2x}{125} $$

for $x \in (0, 5)$. Furthermore, $f_X(x) = 0$ for $x \notin (0, 5)$. Also,

$$ f_Y(y) = \int_{0}^{5} f(x, y) \, dx = \int_{0}^{5} \frac{2(20 - x - 2y)}{625} \, dx = \frac{2[20x - \frac{x^2}{2} - 2xy|_{0}^{5}]}{625} = \frac{7}{25} - \frac{4y}{125}. $$
for \( y \in (0,5) \). Furthermore, \( f_Y(y) = 0 \) for \( y \notin (0,5) \).

d. Are ethanol concentration and acidity independent?

The ethanol concentration and acidity are independent if and only if \( f(x, y) = f_X(x)f_Y(y) \).

We see this is not the case, because \( f_X(x)f_Y(y) = \frac{150-10x}{625} \frac{175-20y}{625} \). Without computing the product we can see that this is not equal to \( f(x, y) = \frac{20-x-2y}{625} \) because it contains the cross-term \( 300xy \). So \( X \) and \( Y \) are not independent.

e. What are the expectation and the variance of the ethanol concentration?

The expectation of \( X \) is
\[
E(X) = \int_{0}^{5} x (\frac{6}{25} - 2x/125) \, dx = \left[ \frac{6x^2}{50} - \frac{2x^3}{375} \right]_{0}^{5} = 3 - \frac{2}{3} = \frac{7}{3}
\]
and the second moment is
\[
E(X^2) = \int_{0}^{5} x^2 (\frac{6}{25} - 2x/125) \, dx = \left[ \frac{2x^3}{25} - \frac{x^4}{250} \right]_{0}^{5} = 10 - \frac{5}{2} = \frac{15}{2}.
\]

Hence the variance is
\[
V(X) = E(X^2) - E(X)^2 = \frac{15}{2} - \frac{49}{18} = \frac{37}{18}.
\]

f. What are the expectation and the variance of the acidity?

\[
E(Y) = \int_{0}^{5} y (\frac{7}{25} - 4y/125) \, dy = \left[ \frac{7y^2}{50} - \frac{4y^3}{375} \right]_{0}^{5} = \frac{7}{2} - \frac{4}{3} = \frac{13}{6}
\]
and
\[
E(Y^2) = \int_{0}^{5} y^2 (\frac{7}{25} - 4y/125) \, dy = \left[ \frac{7y^3}{75} - \frac{y^4}{125} \right]_{0}^{5} = \frac{35}{3} - 5 = \frac{20}{3}.
\]

Hence
\[
V(Y) = E(Y^2) - E(Y)^2 = \frac{20}{3} - \frac{169}{36} = \frac{71}{36}.
\]

g. If the ethanol concentration is 3, what is the conditional probability density function of the acidity?

\[
f_Y|X=3(y) = \frac{f(3,y)}{f_X(3)} = \frac{2(17-2y)}{60}.
\]

h. What is the covariance between ethanol concentration and the acidity?

\[
We previously showed that \( A = \frac{2}{625} \), and that \( E(X) = \frac{7}{3} \) and \( E(Y) = \frac{13}{6} \).

Then \( E(XY) = \int xyf(x,y) \, dx \, dy = \int_{0}^{5} \int_{0}^{5} Axy(20-x-2y) \, dx \, dy \). Integrating with respect to \( x \), \( E(XY) = \int_{0}^{5} yA \left( 10x^2 - x^3 - x^2y \right)_{0}^{5} \, dy \). This simplifies to
\[
E(XY) = \int_{0}^{5} yA \left( \frac{625}{3} - 25y \right) \, dy.
\]

Integrating with respect to \( y \),
\[
E(XY) = A \left( \frac{625}{6} y^2 - \frac{25}{3} y^3 \right)_{0}^{5} = \frac{2}{625} \left( \frac{625 \times 25}{6} - 5 \times 625 \right) = 5.
\]

Then \( \text{Cov}(X,Y) = E(XY) - E(X)E(Y) = 5 - 91/18 = -1/18 \).