1. Question 3 (DeGroot), page 169. Suppose that $X_1$ and $X_2$ have a continuous joint distribution for which the joint probability density function is as follows:

$$f(x_1, x_2) = \begin{cases} 
  x_1 + x_2 & \text{for } 0 < x_1 < 1 \text{ and } 0 < x_2 < 1, \\
  0 & \text{otherwise}.
\end{cases}$$

Find the probability density function of $Y = X_1X_2$.

Set $z = x_2$. The forward relation is $y = x_1x_2$ and $z = x_2$, and the inverse is $x_2 = z$ and $x_1 = y/z$. Then $\frac{dx_1}{dy} = 1/z$, $\frac{dx_2}{dy} = 0$, $\frac{dx_1}{dz} = -y/z^2$, and $\frac{dx_2}{dz} = 1$, and hence $J = 1/z$. The joint density is $1 + y/z^2$ for $z \in (0, 1)$ and $y \in (0, z)$, or for $y \in (0, 1)$ and $z \in (y, 1)$. Then the marginal density for $Y$ is

$$f_Y(y) = \begin{cases} 
  \int_y^1 (1 + y/z^2) \, dz = (1 - y - yz^{-1})_y^1 = 2 - 2y & \text{if } y \in (0, 1) \\
  0 & \text{otherwise}.
\end{cases}$$

2. Question 8 (DeGroot), page 194. Suppose that the proportion of defective items in a large lot is $p$, and suppose that a random sample of $n$ items is selected from the lot. Let $X$ denote the number of defective items in the sample, and let $Y$ denote the number of non-defective items. Find $E(X - Y)$.

Using the summability of expectations, $E(X) = np$, and $Y = n - X$, so

$$E(X - Y) = E(2X - n) = 2E(X) - n = n(2p - 1).$$

3. Question 7 (DeGroot), page 186. Suppose that $X$ and $Y$ have a continuous joint distribution for which the joint probability density function is as follows:

$$f_{X,Y}(x, y) = \begin{cases} 
  12y^2 & \text{for } 0 \leq y \leq x \leq 1, \\
  0 & \text{otherwise}.
\end{cases}$$

Find the value of $E(XY)$.

By definition,

$$E(XY) = \int_0^1 \int_0^x xy12y^2 \, dy \, dx = 12x \int_0^x y^3 \, dy \, dx = \frac{12x}{4}x^4 \left|_0^1 \right. = \frac{3}{2}.$$

Performing the integration with respect to $y$,

$$E(XY) = \int_0^1 3x^5 \, dx = 3 \frac{x^6}{6} \bigg|_0^1 = \frac{1}{2}.$$

In contrast,

$$E(X) = \int_0^1 \int_0^x 12y^2 \, dy \, dx = 12x \int_0^x y^2 \, dy \, dx = \frac{12x}{3}x^3 \bigg|_0^1 = \frac{4}{3}$$

and

$$E(X) = 4 \int_0^1 x^4 \, dx = 4 \frac{x^5}{5} \bigg|_0^1 = 4/5.$$
and

\[ E(Y) = \int_0^1 \int_0^x y12y^2 \, dy \, dx = \int_0^1 12 \int_0^x y^3 \, dy \, dx = \int_0^1 12x^4/4 \, dx \]

and

\[ E(Y) = \int_0^1 3x^4 \, dx = 3 \frac{x^5}{5} \bigg|_0^1 = 3/4. \]

Note that \( E(XY) \neq E(X)E(Y) \). We do not expect equality, because \( X \) and \( Y \) are not independent.