1. Question 6 (DeGroot), page 170. Suppose that $X_1$ and $X_2$ are i.i.d. random variables and that the probability density function of each of them is as follows:

$$f(x) = \begin{cases} 
\exp(-x) & \text{for } x > 0, \\
0 & \text{otherwise.}
\end{cases}$$

Find the probability density function of $Y = X_1 - X_2$.

Let $Z = -X_2$, with density $\exp(z)$ on $(-\infty, 0)$. Then $Y = X_1 + Z$.

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X_1}(y - w) f_Z(w) \, dw.$$ Then $0 \geq w$, and $y \geq w$. Take $y \geq 0$.

**Fig. 1: Integration Region for CDF for Difference of Exponentials**

The convolution integral is

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X_1}(y - w) f_Z(w) \, dw = \int_{-\infty}^{0} \exp(w - y) \exp(w) \, dw.$$  

Factoring out the factor involving $y$,

$$f_Y(y) = \exp(-y) \int_{-\infty}^{0} \exp(2w) \, dw = \exp(-y)/2.$$  

There are two cases, depending on the sign of $y$. For the first case, consider $y \leq 0$.
Then $f_Y(y) = \int_{-\infty}^{y} f_{X_1}(y - w) f_Z(w) \, dw = \int_{-\infty}^{y} \exp(w - y) \exp(w) \, dw = \exp(-y) \exp(2y)/2$.

In either case, the density is $\exp(-|y|)/2$, the Laplace or Double Exponential distribution.