1. Question 5 (DeGroot), page 385. When the motion of a microscopic particle in a liquid or a gas is observed, it is seen that the motion is irregular because the particle collides frequently with other particles. The probability model for this motion, which is called Brownian motion, is as follows: A coordinate system is chosen in the liquid or gas. Suppose that the particle is at the origin of this coordinate system at time \( t = 0 \); and let \( (X, Y, Z) \) denote the coordinates of the particle at any time \( t > 0 \). The random variables \( X \), \( Y \), and \( Z \) are iid, and each of them has a normal distribution with mean 0 and variance \( \sigma^2 t \). Find the probability that at time \( t = 2 \) the particle will lie within a sphere whose center is at the origin and whose radius is \( 4\sigma \).

Let \( \rho = P\left( \sqrt{X^2 + Y^2 + Z^2} < 4\sigma \right) \). First remove the square root to put variables on a chi-square scale:

\[
\rho = P\left( X^2 + Y^2 + Z^2 < 16\sigma^2 \right).
\]

Now rescale to give each variance one:

\[
\rho = P\left( \left( \frac{X}{\sqrt{2}\sigma} \right)^2 + \left( \frac{Y}{\sqrt{2}\sigma} \right)^2 + \left( \frac{Z}{\sqrt{2}\sigma} \right)^2 < 8 \right).
\]

Note that \( (X/(\sqrt{2}\sigma))^2 + (Y/(\sqrt{2}\sigma))^2 + (Z/(\sqrt{2}\sigma))^2 \sim \chi^2_3 \), and the probability is \( \text{pchisq}(8,3) \), which gives 0.954.