Lecture 7

h. Geometric distribution requires an underlying uncountable sample space.
   
   i. Recall set of all infinite success-failure strings is uncountable.
   
   ii. Probability calculations were performed relying on the fact that heuristically the event \( \{N(s) \leq n\} \) can be expressed in terms of strings with \( n \) components.
   
   iii. This smaller set of points is not only countable, but finite.
   
   iv. Probabilities are calculated on these smaller sample spaces.
   
   v. Questions that probabilists know the answer to, but you don’t yet:
      
      • Is there a way to coherently extend probabilities from this restricted set of finite strings to the whole space? Yes.
      
      • Is there more than one way to do this? No.

   WMS: 3.6

5. Negative Binomial Distribution \( \text{NBin}(k, \pi) \)
   
   a. Generalization of geometric distribution.
   
   b. Observe trials yielding either success or failure (like a coin flip)
      
      i. each with the same probability \( \pi \) of yielding success,
      
      ii. until \( k \) successes are observed.
iii. $\pi > 0$ or success will never happen.

c. Let number of trials needed be random variable $N$.

d. What is the probability of seeing success $k$ on the $n$ trial?

i. This can only result from $n - k$ failures and $k - 1$ success in any order followed by one success.

ii. There are $\binom{n-1}{k-1}$ such sequences.

e. Hence the probability of success $k$ on trial $n$ is

$$
\binom{n-1}{k-1} \pi^{k-1} (1 - \pi)^{n-k} \pi = \binom{n-1}{k-1} \pi^k (1 - \pi)^{n-k}.
$$

f. Expectation: $E(N) = \sum_{n=k}^{\infty} n \binom{n-1}{k-1} \pi^k (1 - \pi)^{n-k}$.

i. Algebraic problem: Make this look like a sum that we already know, times a constant.

- Solution is to incorporate the $n$ into the $(n - 1)!$.

ii. Note $n \binom{n-1}{k-1} = \frac{n(n-1)!}{(k-1)!(n-k)!} = k \frac{n!}{k!(n-k)!}.$

iii. So $E(N) = k \sum_{n=k}^{\infty} \binom{n}{k} \pi^k (1 - \pi)^{n-k}$.

iv. Setting $l = k + 1$, $m = n + 1$,

$$E(N) = \frac{k}{\pi} \sum_{m=l}^{\infty} \left( \frac{m-1}{l-1} \right) \pi^{l-1}(1 - \pi)^{m-l} = \frac{k}{\pi}.$$ 

- Sum is the sum of probabilities for the $k + 1$ negative binomial, and is 1.

g. Geometric is special case of negative binomial with $k = 1$. 
i. If that’s so, why did we need the derivative craziness for the geometric, but not for the negative binomial?

ii. Negative binomial has the advantage of being able to compare the sum in the expected value to the probability sum for the next higher distribution.

h. Variance $V(N) = E(N^2) - E(N)^2$.

i. “Follow your nose” approach leads to $E(N^2) = \sum_{n=k}^{\infty} n^2 \binom{n-1}{k-1} \pi^k (1-\pi)^{n-k}$.

- We would instead like to calculate an expectation that puts all of the $n$ stuff into a single easy package.

ii. $E(N(N+1)) = \sum_{n=k}^{\infty} (n+1)n \binom{n-1}{k-1} \pi^k (1-\pi)^{n-k}$.

iii. Note $(n+1)n \binom{n-1}{k-1} = \frac{(n+1)n(n-1)!}{(k-1)!(n-k)!} = k(k+1)\frac{(n+1)!}{(k+1)!(n-k)!}$.

iv. Setting $l = k + 2$, $m = n + 2$, $E(N(N+1)) = (k(k+1)/\pi^2) \sum_{m=l}^{\infty} \binom{m-1}{l-1} \pi^l (1-\pi)^{m-l} = (k(k+1)/\pi^2)$.

v. $E(N^2) = E(N(N+2)) - E(N) = (k(k+1)/\pi^2) - k/\pi$

vi. $V(N) = (k(k+1)/\pi^2) - k/\pi - k^2/\pi^2 = k(1/\pi^2 - 1/\pi)$

i. Calculate CDF by $\text{pnbinom}(x-k,k,p)$.

i. R definition is different: R counts number of failures before success $k$, rather than the total number of trials.
6. Hypergeometric distribution:
   a. An urn is filled with $r$ red and $N - r$ green tickets.
      i. Numbered 1 through $r$ and $r + 1$ through $N$ respectively.
   b. Draw $m$ without replacement.
   c. $X$ is the number of red tickets in sample.
      i. Draws are no longer independent: negative correlation.
      ii. Extreme results are less common and moderate results more common than for binomial.
   d. There are $\binom{N}{m}$ equally-likely sets of drawn tickets.
   e. There are $\binom{r}{x}$ selections of $x$ red tickets to be drawn.
   f. There are $\binom{N-r}{m-x}$ selections of $x$ green tickets to be drawn to give $m - x$ in the sample.
   g. Hence the number of sets of tickets giving $x$ red is the product of these.
   h. Hence $P(X = x) = \binom{r}{x} \binom{N-r}{m-x} / \binom{N}{m}$.
   i. In symbols, Hyper($N, m, r$)
   j. Moments: $E(X) = mr/N$, $V(X) = mr(N - r)(N - m)/(N^2(N - 1))$
i. Expectation is $m\pi$, where $\pi = r/N$.

- $E(X) = \sum x x \binom{r}{x} \binom{N-r}{n-x} / \binom{N}{m}$

- Use the same trick as for the negative binomial:
  - $x$ outside probability cancels with factor in denominator leaving $(x - 1)!$ in denominator.
  - Selectively remove other quantities to make leftovers be sum of another set of hypergeometric probabilities.

ii. Variance is $m(r/N)((N - r)/N)((N - m)/(N - 1)) < m\pi(1 - \pi)$.

- Start with $E(X(X - 1))$ and proceed as above.

k. Express as corner in $2 \times 2$ classification

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$X$</td>
<td>$r - X$</td>
<td>$r$</td>
</tr>
<tr>
<td>$m - X$</td>
<td>$N - r - m + X$</td>
<td>$N - r$</td>
</tr>
<tr>
<td>$m$</td>
<td>$N - m$</td>
<td>$N$</td>
</tr>
</tbody>
</table>

i. Remaining corners are obtained through subtraction.

l. Legitimate $X$ values make all table entries non-negative:

$X \geq 0, N - r - m, X \leq r, m$.

m. As $N \rightarrow \infty$ with $\pi = r/N$ constant, dependency decreases, and distribution behaves in the limit like binomial.

n. CDF $\text{phyper}(x, m, N-m, r) = \text{phyper}(x, r, N-r, m)$
7. Poisson distribution

a. Symbols: \( \text{Pois}(\lambda) \)

b. probability function
\[
p_X(x; \lambda) = \exp(-\lambda)\lambda^x / x! \quad \text{for} \quad \lambda \in [0, \infty).
\]

c. Recall that \( \sum_{x=0}^{\infty} \lambda^x / x! = \exp(\lambda) \), so this is indeed the probability function for a distribution.

d. Expectation as in trick above:
\[
E(X) = \sum_{x=0}^{\infty} x \exp(-\lambda)\lambda^x / x! = \lambda
\]

e. Variance is easier to do by first calculating expectation of something that can cancel with part of factorial

i. Try
\[
E(X(X - 1)) = \sum_{x=0}^{\infty} x(x - 1) \exp(-\lambda)\lambda^x / x! = \lambda^2 + \lambda - \lambda^2 = \lambda
\]
f. In binomial, as $m \to \infty$ s.t. $\lambda = \pi m$ remains constant, then

$$P(X = x) = \frac{\lambda^x}{x!} \frac{m(m-1) \cdots (m-x+1)}{m^x} \left(1 - \frac{\lambda}{m}\right)^m \left(1 - \frac{\lambda}{m}\right)^{-x}$$

i. $\left(1 - \frac{\lambda}{m}\right)^m \to \exp(-\lambda)$

ii. $\frac{m(m-1) \cdots (m-x+1)}{m^x} \to 1$

iii. $\left(1 - \frac{\lambda}{m}\right)^{-x} \to 1$

iv. Hence $P(X = x) \to \frac{\lambda^x \exp(-\lambda)}{x!}$.

g. CDF $\text{ppois}(x, \lambda)$. 