h. Geometric distribution requires an underlying uncountable sample space.
i. Recall set of all infinite success-failure strings is uncountable.
ii. Probability calculations were performed relying on the fact that heuristically the event \( \{ N(s) \leq n \} \) can be expressed in terms of strings with \( n \) components
iii. This smaller set of points is not only countable, but finite.
iv. Probabilities are calculated on these smaller sample spaces.
v. Questions that probabilists know the answer to, but you don’t yet:
   • Is there a way to coherently extend probabilities from this restricted set of finite strings to the whole space? Yes.
   • Is there more than one way to do this? No.
WMS: 3.6

5. Negative Binomial Distribution NBin\((k, \pi)\)
a. Generalization of geometric distribution.
b. Observe trials yielding either success or failure (like a coin flip)
i. each with the same probability \( \pi \) of yielding success,
ii. until \( k \) successes are observed.
iii. \( \pi > 0 \) or success will never happen.
c. Let number of trials needed be random variable \( N \).
d. What is the probability of seeing success \( k \) on the \( n \) trial?

- We would instead like to calculate an expectation that puts all of the \( n \) stuff into a single easy package.
i. \( E(N(N + 1)) = \sum_{n=k}^{\infty} (n+1)n (\frac{n-1}{k-1}) \pi^k (1 - \pi)^{n-k} \).
ii. Note \( (n+1)n = \frac{(n+1)n(n-1)!}{(k-1)!(n-k)!} = k(k+1)/2 \).
iii. Setting \( l = k+2 \), \( m = n+2 \),
   \( E(N(N + 1)) = (k(k+1)/\pi^2) \sum_{m=l}^{\infty} (m-1) \pi^l (1 - \pi)^{m-l} = (k+1)/\pi^2 \).
v. \( E(N^2) = E(N(N + 2)) - E(N) = (k(k+1)/\pi^2) - k/\pi \).
vi. \( V(N) = (k(k+1)/\pi^2) - k/\pi - k^2/\pi^2 = (k/\pi - 1) \pi \).
i. Calculate CDF by \( \text{pbinom}(x-k, k, \pi) \).
   i. R definition is different: \( R \) counts number of failures before success \( k \), rather than the total number of trials.
WMS: 3.7

6. Hypergeometric distn:
a. An urn is filled with \( r \) red and \( N-r \) green tickets.
i. Numbered 1 through \( r \) and \( r+1 \) through \( N \) respectively.
b. Draw \( m \) without replacement.
c. \( X \) is the number of red tickets in sample.
i. Draws are no longer independent: negative correlation.
ii. Extreme results are less common and moderate results more common than for binomial.
d. There are \( \binom{N}{m} \) equally-likely sets of drawn tickets.
e. There are \( \binom{r}{x} \) selections of \( x \) red tickets to be drawn.
f. There are \( \binom{N-r}{m-x} \) selections of \( x \) green tickets tickets to be drawn to give \( m-x \) in the sample.
g. Hence the number of sets of tickets giving \( x \) red is the product of these.
h. Hence \( P(X = x) = \binom{r}{x} \binom{N-r}{m-x} / \binom{N}{m} \).
i. In symbols, \( \text{Hyper}(N,m,r) \).
j. Moments: \( E(X) = mr/N \),
   \( V(X) = m(r/m)(N-m)/(N^2(N-1)) \).
i. Expectation is \( mr \pi \), where \( \pi = r/N \).
   • \( E(X) = \sum_{x} x \binom{r}{x} \binom{N-r}{m-x} / \binom{N}{m} \).
   • Use the same trick as for the negative binomial:
     - \( x \) outside probability cancels with factor in denominator leaving \( (x-1)! \) in denominator.
     - Selectively remove other quantities to make leftovers be sum of another set of hypergeometric probabilities.
ii. Variance is \( m(r/m)(N-m)/(N^2(N-1)) \) < \( m \pi (1 - \pi) \).
   • Start with \( E(X(X-1)) \) and proceed as above.
k. Express as corner in \( 2 \times 2 \) classification
   \[
   \begin{array}{c|c|c}
   X & r-X & \hline
   m-X & N-r-m+x & \pi
   \end{array}
   \]
   \[
   \begin{array}{c|c|c}
   m & N-r & \hline
   m & N-m & \pi
   \end{array}
   \]
i. Remaining corners are obtained through subtraction.
Lecture 7

1. Legitimate $X$ values make all table entries non-negative: $X \geq 0, N - r - m, X \leq r, m$.

2. As $N \to \infty$ with $\pi = r/N$ constant, dependency decreases, and distribution behaves in the limit like binomial.

3. CDF $\text{phyper}(x, m, N-m, r) = \text{phyper}(x, r, N-r, m)$

WMS: 3.8

7. Poisson distribution

a. Symbols: $\text{Pois}(\lambda)$

b. Probability function $p_X(x; \lambda) = \exp(-\lambda)\lambda^x/x!$ for $\lambda \in [0, \infty)$.

c. Recall that $\sum_{x=0}^{\infty} \frac{\lambda^x}{x!} = \exp(\lambda)$, so this is indeed the probability function for a distribution.

d. Expectation as in trick above:

$$E(X) = \sum_{x=0}^{\infty} x \exp(-\lambda)\lambda^x/x!$$

$$= \lambda \sum_{x=1}^{\infty} \exp(-\lambda)\lambda^{x-1}/(x-1)! = \lambda$$

e. Variance is easier to do by first calculating expectation of something that can cancel with part of factorial

i. Try

$$E(X(X-1)) = \sum_{x=0}^{\infty} x(x-1) \exp(-\lambda)\lambda^x/x!$$

$$= \lambda^2 \sum_{x=2}^{\infty} \exp(-\lambda)\lambda^{x-2}/(x-2)! = \lambda^2$$

$$\text{Var}(X) = \lambda^2 + \lambda - \lambda^2 = \lambda$$

f. In binomial, as $m \to \infty$ s.t. $\lambda = \pi m$ remains constant, then

$$P(X = x) = \frac{m!}{x! (m-x)!} \pi^x (1-\pi)^{m-x}$$

$$= \lambda^x m(m-1) \cdots (m-x+1) \left( 1 - \frac{\lambda}{m} \right)^m \left( 1 - \frac{\lambda}{m} \right)^{-x}$$

i. $(1 - \frac{\lambda}{m})^m \to \exp(-\lambda)$

ii. $\frac{m(m-1) \cdots (m-x+1)}{m^x} \to 1$

iii. $(1 - \frac{\lambda}{m})^{-x} \to 1$

iv. Hence $P(X = x) \to \frac{\lambda^x \exp(-\lambda)}{x!}$.

g. CDF $\text{ppois}(x, \lambda)$.

This page intentionally left blank.