5. Summation Tricks:
   a. For expectation: \( E(X) = \sum_x x p_X(x) \)
      i. For probabilities involving factorial of random variable value
         - Incorporate random variable value into the factorial.
         - If the factorial is in the numerator, move argument up by one: Negative binomial
         - If the factorial is in the denominator, move argument down by one: Binomial, Poisson
      ii. Re-parameterize the remainder of the quantities
      iii. Identify sum recognizable as summing to 1
   b. Second Moment:
      i. Sometimes is easier to calculate expectation of \( X(X - 1) \) if factorial is in denominator.
      ii. Or \( X(X + 1) \) if factorial is in numerator.
   c. Moment and Probability Generating Function
      \( E(e^{tx}) \) and \( E(e^{x^2}) \)

4. The density is the derivative of the distribution function:
   a. \( f_X(x) = \frac{d}{dx} F_X(x) \)
   i. These involve the expectation of a quantity raised to power \( X \)
      - \( t \) for probability generating function
      - \( \exp(t) \) for mgf
   ii. Most of the probability functions under consideration involve a quantity raised to the power \( x \)
   iii. Multiply these to get a single quantity raised to the power \( x \).
   iv. Recognize this as the probability function for a distribution of the same form with a different parameter.

WMS: 3.11

6. Probability Inequalities
   a. Markov Inequality: If \( P(X \geq 0) = 1 \), then for all \( t > 0 \), \( P(X \geq t) \leq E(X)/t \).
   i. Марков (Markoff)
   ii. Proof: Split sum according to whether \( x \) is at least as great as \( t \).
      \[ E(X) = \sum_x x p_X(x) = \sum_{x < t} x p_X(x) + \sum_{x \geq t} x p_X(x) \]
      - Part with lower values is positive, by positivity; throw it away.
      \[ E(X) \geq \sum_{x \geq t} x p_X(x) \]
      - Bound sum below by lower bound on \( x \):
      \[ E(X) \geq \sum_{x \geq t} \frac{1}{t} p_X(x) \]
      - Factor out \( t \): \( E(X)/t \geq \sum_{x \geq t} p_X(x) \).
   b. Tchebycheff Inequality:
      \[ P \left( (X - E(X))^2 \geq t \right) \leq V(X)/t \]
      - as can be seen by applying Markov’s inequality to \( (X - E(X))^2 \).
   i. Чебышёв (Chebycheff, Chebychov, Chebyshov; or Tchebychev, Tchebycheff (French transcriptions); or Tschebyschev, Tschebyschef, Tschebyscheff (German transcriptions); Čebyčev. (from Wikipedia).

We can use Tchebycheff inequality to show \( V(X) = 0 \) if and only if there exists \( c \) such that \( P(X = c) = 1 \).
   i. Suppose \( P(X = c) = 1 \).
      - Then \( E(X) = c \).
      - Also \( V(X) = \sum_{x \neq c} (x - c)^2 p_X(x) = 0 \).
   ii. Suppose \( V(X) = 0 \).
      - By Tchebycheff inequality,
        \[ P \left( |X - E(X)| \geq t \right) \leq V(X)/t^2 = 0 \]
        for all \( t > 0 \).
      - Let \( A = \{X \neq E(X)\} = \{|X - E(X)| > 0\} \)
      - Let \( A_n = \{|X - E(X)| > 1/n\} \).
      - Then \( A = \bigcup_{n=1}^{\infty} A_n \).
      - Then \( P(A) \leq \sum_{n=1}^{\infty} P(A_n) = 0 \).

WMS: 4.1

IV. Continuous Distributions

A. Introduction to Continuous Distributions
   1. Definition and Examples
      a. Recall distribution function \( F_X(x) = P(X \leq x) \).
      b. A continuous distri is one in which the distribution function is given an integral \( F_X(x) = \int_{-\infty}^{x} f_X(y) \, dy \).
      i. To calculate the probability associated with a particular range of values, integrate a function called
         a probability density function
         over the range.
      c. Such a variable could take on any value in a range of real numbers.
      d. Examples

WMS: 4.2

4. The density is the derivative of the distribution function.
   a. \( dF_X(x_1)/dx_1 = f_X(x_1) \), if \( f_X \) is continuous at \( x_1 \).
   i. Take:

WMS: 85 Lecture 9 86 Lecture 9 87 Lecture 9 88
• $x_2 > x_1$
• and if $x_2$ is close to $x_1$,
ii. Then
  • $F_X(x_2) - F_X(x_1) = P(x_1 < X \leq x_2) = \int_{x_1}^{x_2} f_X(y) \, dy \approx (x_2 - x_1) f_X(x_1)$
  • Hence $(F_X(x_2) - F_X(x_1))/(x_2 - x_1) \approx f_X(x_1)$
  • Hence $dF_X(x_1)/dx_1 = f_X(x_1)$: Fundamental theorem of calculus.

b. Uniform Example:
  i. $f_X(x) = 1$ for $x \in (0, 1)$, and equal 0 elsewhere.
  • Called “Uniform on $(0, 1)$”.
  ii. Then
    
    $F_X(x) = \begin{cases} 
    0 & \text{if } x \leq 0 \\
    x & \text{if } x \in (0, 1) \\
    1 & \text{if } x \geq 1 
    \end{cases}$

  iii. Then
    
    $F_X'(x) = \begin{cases} 
    0 & \text{if } x < 0 \\
    1 & \text{if } 1 \in (0, 1) \\
    0 & \text{if } x > 1 \\
    \text{undefined} & \text{if } x \in \{0, 1\} 
    \end{cases}$