iv. \( X_1, \ldots, X_k \sim \text{NBin}(\theta, m) \)

- Number of trials it takes to get \( m \) successes, if each has success probability \( \theta \)
- Likelihood for one observation \( L(\theta) = \frac{X-1}{m-1}\theta^m(1-\theta)^{X-m} \)
- Log likelihood for one observation \( \ln\left(\frac{X-1}{m-1}\right) + m \ln(\theta) + (X - m) \ln(1 - \theta) \)
- Overall log likelihood \( \sum_{j=1}^{n}[\ln\left(\frac{X_j-1}{m-1}\right) + m \ln(\theta) + (X_j - m) \ln(1 - \theta)] = \sum_{j=1}^{n}\ln\left(\frac{X_j-1}{m-1}\right) + nm \ln(\theta) + (\sum_{j=1}^{n} X_j - nm) \ln(1 - \theta) \)
- \( l'(\theta) = nm/\theta - (\sum_{j=1}^{n} X_j - nm)/(1 - \theta) \)
- MLE satisfies \( nm\hat{\theta} - (\sum_{j=1}^{n} X_j - nm)/(1 - \hat{\theta}) = 0 \), \( nm - nm\hat{\theta} - (\sum_{j=1}^{n} X_j - nm)\hat{\theta} = 0 \), \( nm - \sum_{j=1}^{n} X_j \hat{\theta} = 0 \), \( \hat{\theta} = m/\bar{X} \).
- By invariance, \( \hat{\mu} = \bar{X} \), unbiased.

F: 10.8

v. Bivariate Normal

- \( X_i \sim \mathcal{N}(0, 1) \), \( Y_i | X_i \sim \mathcal{N}(\rho X_i, 1 - \rho^2) \)
Lecture 8

• \( l(\rho) = \)
\[
\sum_{j=1}^{n} \left[ -\frac{X_j^2}{2} - \frac{(Y_j - \rho X_j)^2}{2(1 - \rho^2)} - \ln(2\pi) - \frac{1}{2} \ln(1 - \rho^2) \right]
\]

• \( l'(\rho) = \)
\[
\sum_{j=1}^{n} \left[ -\frac{2(1 - \rho^2)(Y_j - \rho X_j)2(-X_j) - (Y_j - \rho X_j)^24\rho}{4(1 - \rho^2)^2} \right.
\]
\[
+ \frac{\rho}{(1 - \rho^2)} \]
\[
= \sum_{j=1}^{n} \left[ \frac{(1 - \rho^2)(Y_j - \rho X_j)X_j - (Y_j^2 - 2\rho X_j Y_j + \rho^2 X_j^2)\rho}{(1 - \rho^2)^2} \right.
\]
\[
+ \frac{\rho}{(1 - \rho^2)} \]
\[
= \frac{(1 - \rho^2)(S_{xy} - \rho S_{xx}) - (S_{yy} - 2\rho S_{xy} + \rho^2 S_{xx})\rho}{(1 - \rho^2)^2}
\]
\[
+ \frac{\rho}{(1 - \rho^2)n}
\]

• Gives cubic equation in \( \rho \)

vi. Weibull

• \( X_1, \ldots, X_k \sim \exp(-x^\alpha/\theta)\theta^{-1}x^{\alpha-1}, \alpha \text{ known} \)

• \( X_j^\alpha \sim \mathcal{E}(\theta) \)

• MLE \( \hat{\theta} = \sum_{j=1}^{n} \frac{X_j^\alpha}{n} \)

• \( \text{E} [X_j] = \Gamma(1 + 1/\alpha)\theta^{1/\alpha} \)

• m.o.m.e. satisfies \( \bar{X} = \Gamma(1 + 1/\alpha)\theta^{1/\alpha} \) if and only if
\[ \hat{\theta} = \left( \bar{X} / \Gamma\left(1 + 1/\alpha\right) \right)^\alpha \]

h. Rules for function maximization:
   i. Taking the derivative(s) of the likelihood function.
   ii. Solving the equation(s) derived by setting these \( = 0 \).
   iii. Ensuring that we have a local maximum, possibly by checking to see that the second derivative \( < 0 \) at the proposed maximum,
   iv. Ensuring that our local max is a global max, possibly by checking
      - that either \( L''(\theta) < 0 \) ∀\( \theta \)
   i. Properties of Maximum Likelihood estimates?
      i. Are they unbiased? No, but almost...
      ii. Are they sufficient? No, but almost...
      iii. Are they consistent? No, but almost...
      iv. Are they efficient? No, but almost...
   j. For ind. observations,
      \[ f_{X_1, \ldots, X_n}(x_1, \ldots, x_n; \theta) = \prod_{1}^{n} f_{X_j}(x_j; \theta) \]
and hence

\[ L(\theta; X_1, \ldots, X_n) = \prod_{1}^{n} L(\theta; X_j) \]

and

\[ l(\theta; X_1, \ldots, X_n) = \sum_{j=1}^{n} l(\theta; X_j) \]

so the log likelihood for a collection of ind. random variables is the sum of the ind. likelihoods.

4. In general, m.o.m.e. = m.l.e. if density or mass function is of form

\[ \exp(c(\theta)x + b(\theta) + d(x)) \]
<table>
<thead>
<tr>
<th>Distribution</th>
<th>Density</th>
<th>m.o.m.e.</th>
<th>m.l.e.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bin($m, \theta$)</td>
<td>$\binom{m}{x} \theta^x (1 - \theta)^m$</td>
<td>$\hat{\theta} = \bar{X}/m$</td>
<td></td>
</tr>
<tr>
<td>NBin($m, \theta$)</td>
<td>$\binom{x-1}{m-1} \theta^x (1 - \theta)^m$</td>
<td>$\hat{\theta} = m/\bar{X}$</td>
<td></td>
</tr>
<tr>
<td>$\mathcal{P}(\theta)$</td>
<td>$\exp(-\theta) \theta^x / x!$</td>
<td>$\hat{\theta} = \bar{X}$</td>
<td></td>
</tr>
<tr>
<td>$\mathcal{N}(\mu, \sigma^2)$</td>
<td>$\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$</td>
<td>$\hat{\mu} = \bar{X}$, $\hat{\sigma} = \sqrt{\frac{\sum_{j=1}^{n} (X_j - \bar{X})^2}{n}}$</td>
<td></td>
</tr>
<tr>
<td>$\mathcal{U}(0, \theta)$</td>
<td>$\begin{cases} 1 &amp; \text{if } x \in [0, \theta] \ 0 &amp; \text{ow} \end{cases}$</td>
<td>$\hat{\theta} = 2\bar{X}$</td>
<td>$\hat{\theta} = \max(X_j)$</td>
</tr>
<tr>
<td>$\mathcal{C}(\theta)$</td>
<td>$\frac{1}{\pi(1+(x-\theta)^2)}$</td>
<td>Does’t exist</td>
<td>Exists; no closed form expression</td>
</tr>
<tr>
<td>$\mathcal{W}(\alpha, \theta)$</td>
<td>$\exp\left(-\frac{x^\alpha}{\theta}\right) \left(\frac{\alpha}{x}\right) \times x^{\alpha-1}$</td>
<td>$\hat{\theta} = \frac{\sum_{j=1}^{n} X_j^{1/\alpha}}{n}$</td>
<td>$\hat{\theta} = \left(\frac{\bar{X}}{\Gamma(1+\frac{1}{\alpha})}\right)^\alpha$</td>
</tr>
</tbody>
</table>