iii. Case 2: $\sigma = \tau$, but common value is unknown.

- Estimate common value by

$$s_p = \sqrt{\frac{\sum_{j=1}^{n}(X_j - \bar{X})^2 + \sum_{j=1}^{m}(Y_j - \bar{Y})^2}{n + m - 2}}$$

- Homework shows that the result is $\sim T(n + m - 2)$

- Hence relation between degrees of freedom and number of observations is more complicated than before.

- Hence CI is $\bar{Y} - \bar{X} \pm s_p \sqrt{1/n + 1/mt_{n+m-2,\alpha/2}} = \bar{Y} - \bar{X} \pm s_p \sqrt{1/n + 1/mt_{n+m-2,\alpha/2}}$

iv. Case 3: $\sigma$ and $\tau$ are unknown.

- Estimate separately using usual formulae.

- $T = \frac{\bar{Y} - \bar{X} - (\nu - \mu)}{\sqrt{[\hat{\sigma}^2/(n - 1)]/n + [\hat{\tau}^2]/m}} \sim ?$

- ? depends on relation between $\sigma$ and $\tau$

  - Best case: $\sigma = \tau$ $\Rightarrow$ DF almost $n + m - 2$

  - Worst case: If $n \geq m$, then $\sigma = 0$ $\Rightarrow$ $T(m - 1)$

  - Usual solution: complicated combination of $\hat{\sigma}$ and $\hat{\tau}$.

- Hence CI is $\bar{Y} - \bar{X} \pm$
\[ \sqrt{\frac{\sum_{j=1}^{n}(X_j - \bar{X})^2/(n - 1)}{n} + \frac{\sum_{j=1}^{m}(Y_j - \bar{Y})^2/(m - 1)}{mt^2, \alpha/2}} \]

F: 11.4

e. Example: Binomial Distribution: \( X \sim \text{Bin}(\theta, m) \).

i. can we create a pivotal quantity?

* No.

ii. For small samples confidence intervals may be calculated exactly. The accompanying figures demonstrate construction of exact confidence intervals for small binomial problems.

* \( \forall \theta \in (0, 1) \),

  \[ \forall q \in [0, 1] \text{ calculate } P_\theta [Q \leq q] \text{ for } Q = X/m \]

  \[ \text{select } \max\{q | P_\theta [Q \leq q] \geq 1 - \alpha/2 \}. \]

  \[ \forall q \in [0, 1] \text{ calculate } P [Q \geq \theta]. \]

  \[ \text{select } \min\{q | P_\theta [Q \geq q] \geq 1 - \alpha/2 \}. \]

* Confidence interval is the region between extreme endpoints for segments.

iii. For larger \( n \), approximations make things easier:
Confidence Construction for Binomial Variables

\( \hat{\pi} \)

\( \pi \)

3 trials
Confidence Construction for Binomial Variables

\[ \hat{\pi} \]

\[ \pi \]

3 trials

Exact
Hard Approx
Easy Approx
• Approximate pivotal quantity: 
\[
\frac{Q - \theta}{\sqrt{\theta(1 - \theta)/m}} \sim N(0, 1).
\]

\[
P \left[ z_{\alpha/2} \leq \frac{Q - \theta}{\sqrt{\theta(1 - \theta)/m}} \leq z_{1-\alpha/2} \right] = 1 - \alpha,
\]

• Confidence interval is that set of \( \theta \) such that 
\[-z_{\alpha/2} \leq \frac{\sqrt{m(Q - \theta)}}{\sqrt{\theta(1 - \theta)}} \leq z_{1-\alpha/2} \]
holds; or equivalently, that set where 
\[
\frac{(Q - \theta)^2}{\theta(1 - \theta)/m} \leq z_{\alpha/2}^2.
\]

• Solve inequality \((Q - \theta)^2 \leq z_{\alpha/2}^2(\theta(1 - \theta)/m)\)