8. Inference about two normal populations, equal variance:
   a. \( X_1, \ldots, X_m \sim \mathcal{N}(\mu, \sigma^2) \) and \( Y_1, \ldots, Y_n \sim \mathcal{N}(\nu, \tau^2) \) and all are ind..
   b. log likelihood

\[
l(\mu, \nu, \sigma, \tau) = -\frac{\sum j (X_j - \mu)^2}{2\sigma^2} - \frac{\sum j (Y_j - \nu)^2}{2\tau^2} - m \log(\sigma) - n \log(\tau).
\]

c. Assume \( \sigma = \tau \).

d. m.l.e. for \( \mu \) and \( \nu \) under \( H_A \): \( \bar{X} \) and \( \bar{Y} \).

e. m.l.e. for \( \mu \) and \( \nu \) under \( H_0 \):

\[
\bar{Z} = \frac{\sum_j X_j + \sum_j Y_j}{m + n}.
\]

f. \( \hat{\sigma}^2 \) = m.l.e. for \( \sigma^2 \) under \( H_A \) solves

\[
0 = \frac{\sum j (X_j - \bar{X})^2 + \sum j (Y_j - \bar{Y})^2}{\hat{\sigma}^3} - \frac{m + n}{\hat{\sigma}}
\]

\[
\hat{\sigma}^2 = \frac{\sum j (X_j - \bar{X})^2 + \sum j (Y_j - \bar{Y})^2}{m + n}
\]
g. As before m.l.e. for $\sigma^2$ under $H_0$ is

$$\hat{\sigma}^2 = \frac{\sum_j (X_j - \bar{Z})^2 + \sum_j (Y_j - \bar{Z})^2}{m + n},$$

h. g.l.r.t. for $\mu = \nu$ vs $\mu \neq \nu$:

$$\Lambda = \exp \left( -\frac{\sum_j (X_j - \bar{Z})^2}{2\hat{\sigma}^2} - \frac{\sum_j (Y_j - \bar{Z})^2}{2\hat{\sigma}^2} \right) \frac{\hat{\sigma}^{-m-n}}{ \left[ \exp \left( \frac{\sum_j (X_j - \bar{X})^2}{2\hat{\sigma}^2} + \frac{\sum_j (Y_j - \bar{Y})^2}{2\hat{\sigma}^2} \right) \right] ^{-m-n} }$$

$$= \exp \left( -\frac{m + n}{2} \right) \hat{\sigma}^{-m-n} \exp \left( \frac{m + n}{2} \right) \hat{\sigma}^{m+n}$$

$$= (\hat{\sigma} / \tilde{\sigma})^{m+n}$$

i. Test statistic is ratio of powers of sum of squares of deviations from means.

ii. The only such ratios whose distn's we can handle

- are to power 1.

- with sums ind..

iii. Noting that test is equivalent to rejecting when $\hat{\sigma}^2 / \tilde{\sigma}^2$ small goes half way.

iv. Noting that sum of squares about $\bar{Z}$ can be expressed as sum of squares about $\bar{X}$ or $\bar{Y}$ plus an ind. bit does rest.
i. Note

\[
\sum_j (X_j - \bar{Z})^2 = \sum_j (X_j - \bar{X} + \bar{X} - \bar{Z})^2 = \\
\sum_j (X_j - \bar{X})^2 + m(\bar{X} - \bar{Z})^2
\]

ii. \[
\sum_j (Y_j - \bar{Z})^2 = \sum_j (Y_j - \bar{Y})^2 + n(\bar{Y} - \bar{Z})^2.
\]

iii. \[
\bar{Y} - \bar{Z} = \frac{(m + n)\bar{Y} - n\bar{Y} - m\bar{X}}{m + n} = \frac{m(\bar{Y} - \bar{X})}{m + n}.
\]

iv. \[
\bar{X} - \bar{Z} = n(\bar{X} - \bar{Y})/(m + n).
\]

j. Then

\[
\hat{\sigma}^2 - \tilde{\sigma}^2 = n(\bar{Y} - \bar{Z})^2 + m(\bar{X} - \bar{Z})^2 \\
= (m + n)^{-2}[nm^2 + mn^2](\bar{X} - \bar{Y})^2 \\
= (1/m + 1/n)^{-1}(\bar{X} - \bar{Y})^2.
\]

k. Reject when large:

\[
\frac{\tilde{\sigma}^2 - \hat{\sigma}^2}{\hat{\sigma}^2} = \left(\frac{1}{m} + \frac{1}{n}\right)^{-1}(\bar{X} - \bar{Y})^2.
\]

l. Multiplying the test statistic by \(\frac{m + n}{m + n - 2}\) to give right numerator:

\[
\frac{(\bar{X} - \bar{Y})^2}{\left(\frac{1}{m} + \frac{1}{n}\right)S_p^2} \text{ large, for } S_p^2 = \frac{\sum_j (X_j - \bar{X})^2 + \sum_j (Y_j - \bar{Y})^2}{m + n - 2}.
\]

m. Equiv., reject when \(|T|\) large, for \(T = \frac{(\bar{X} - \bar{Y})}{S_p\sqrt{1/m + 1/n}}\).

i. \(T \sim t(m + n - 2)\).

ii. Reject when \(|T| \geq t_{1-\alpha/2}(m + n - 2)\).
9. Inference about two normal means, unequal variance:
   a. $X_1, \ldots, X_m \sim \mathcal{N}(\mu, \sigma^2)$, $Y_1, \ldots, Y_n \sim \mathcal{N}(\nu, \tau^2)$.
   b. This is called the Behrens–Fisher problem.
   c. Hard: there is no pivotal quantity.
   d. Test statistic: $T = (S_x^2/m + S_y^2/n)^{-1/2}(\bar{X} - \bar{Y})$.
   e. $S_x^2$ and $S_y^2$ are usual unbiased variance estimates for $X$’s and $Y$’s separately.
   f. distn of $T$ depends on unknown $\sigma/\tau$.
   g. Often approximated using a $t$ distn
      i. d.f. chosen to approximate well for a wide range of $\sigma/\tau$.
      ii. Welch’s approximation.

10. Inference about normal variance:
    a. $X_1, \ldots, X_m \sim \mathcal{N}(\mu, \sigma^2)$
        i. $H_0 : \sigma = \sigma_0$ vs. $H_A : \sigma \neq \sigma_0$
    b. $\hat{\mu} = \bar{X}$
    c. $\hat{\sigma} = \sigma_0$, $\tilde{\sigma} = \sqrt{\frac{\sum_{j=1}^{m} (X_j - \bar{X})^2}{m}}$
    d. Likelihood ratio statistic is
\[ \Lambda = \frac{\exp \left( -\frac{\sum_{j=1}^{m} (X_j - \bar{X})^2}{2\sigma_0^2} \right) \sigma_0^{-m}}{\exp \left( -\frac{\sum_{j=1}^{m} (X_j - \bar{X})^2}{2 \sum_{j=1}^{m} (X_j - \bar{X})^2 / m} \right) \left( \sum_{j=1}^{m} (X_j - \bar{X})^2 / m \right)^{-m/2}} \]

\[ = \exp \left( -\frac{\sum_{j=1}^{m} (X_j - \bar{X})^2}{2\sigma_0^2} \right) \left( \frac{\sum_{j=1}^{m} (X_j - \bar{X})^2}{\sigma_0^2} \right)^{m/2} \times \exp (m/2) m^{m/2} \]
Likelihood Ratio Test Statistic for Testing a Variance

\[ \Lambda = \sqrt{\sum_{j=1}^{m} (X_j - \bar{X})^2 / \sigma_0^2} \]

Straight lines connect critical values for equal tailed test. Slope of line indicates departure of standard equal-tailed test from generalized likelihood ratio test.