11. Inference about two normal variances:

a. $X_1, \ldots, X_m \sim N(\mu, \sigma^2)$ \quad $Y_1, \ldots, Y_n \sim N(\nu, \tau^2)$

i. $H_0 : \sigma = \tau \text{ vs. } H_A : \sigma \neq \tau$

b. $\hat{\mu} = \tilde{\mu} = \bar{X}$, $\hat{\nu} = \tilde{\nu} = \bar{Y}$

c. $\tilde{\sigma} = \sqrt{\frac{1}{m} \sum_{j=1}^{m} (X_j - \bar{X})^2}$, $\tilde{\tau} = \sqrt{\frac{1}{n} \sum_{j=1}^{n} (Y_j - \bar{Y})^2}$,

\[
\hat{\sigma} = \tilde{\tau} = \sqrt{\left[ \frac{1}{m} \sum_{j=1}^{m} (X_j - \bar{X})^2 + \frac{1}{n} \sum_{j=1}^{n} (Y_j - \bar{Y})^2 \right] / (m + n) =} \sqrt{\gamma \tilde{\sigma}^2 + (1 - \gamma) \tilde{\tau}^2} \text{ for } \gamma = m / (m + n)
\]

d. Likelihood ratio statistic is

\[
\Lambda = \exp \left( -\frac{\sum_{j=1}^{m} (X_j - \bar{X})^2}{2\hat{\sigma}^2} - \frac{\sum_{j=1}^{n} (Y_j - \bar{Y})^2}{2\hat{\tau}^2} \right) \tilde{\sigma}^{-m\tilde{\tau} - n} \frac{\exp \left( -\frac{m+n}{2} \tilde{\sigma}^{-m\tilde{\tau} - n} \right)}{\exp \left( -\frac{m}{2} \tilde{\sigma}^{-m\tilde{\tau} - n} - \frac{n}{2} \tilde{\tau}^{-m\tilde{\tau} - n} \right)} = \frac{\gamma \tilde{\sigma}^2 + (1 - \gamma) \tilde{\tau}^2}{\tilde{\sigma}^{-m\tilde{\tau} - n}} - (m+n)/2
\]

e. When $m = n$, $\Lambda = \left( \frac{1}{2} \tilde{\sigma} / \tilde{\tau} + \frac{1}{2} \tilde{\tau} / \tilde{\sigma} \right)^{-m+n}/2$

i. Hence reject $H_0$ if $\tilde{\sigma}^2 / \tilde{\tau}^2 > c$ or $\tilde{\tau}^2 / \tilde{\sigma}^2 > c$

ii. $\tilde{\tau}^2 / \tilde{\sigma}^2 \sim F_{n-1, m-1}$

f. When $m \neq n$, not exactly right

\[F: 13.8\]
12. Multinomial Inference

a. Extension of Binomial:

i. Draw $m$ items that are either successes or failures.
   - Successes have probability $\pi$
   - Failures have probability $1 - \pi$

ii. Record total # of successes.

b. Multinomial Distribution Definition:

i. Draw $m$ items that fall into one of $J$ groups.
   - Group $j$ has probability $\pi_j$
   - Hence $\sum j \pi_j = 1$

ii. Record:
   - Raw data $Y_1, \ldots, Y_m$, where each $Y_j \in \{1, \ldots, J\}$.
   - Sufficient reduction $X_1, \ldots, X_J$ the #s of successes in each group.
   - Hence $\sum j X_j = m$.

c. Multivariate p.m.f.:

i. for raw data: $P[Y = \mathbf{y}; \boldsymbol{\pi}] = \prod_{j=1}^{m} \pi_{Y_j} = \prod_{j=1}^{J} \pi_l^{X_l}$.

ii. for reduced data:
\[
P[X = x; \pi] = \sum_{y \text{ yielding } x} P[Y = y; \pi]
\]
\[
= \left[ \prod_{l=1}^{J} \frac{x_l^l}{\pi_l^l} \right] \times \# \text{ of } y \text{ associated with } x
\]
\[
= \left[ \prod_{l=1}^{J} \frac{x_l^l}{\pi_l^l} \right] \times \# \text{ of ways to get } x_1 \times \# \text{ of ways to get } x_2 \text{ from remaining } m - x_1 \times \cdots
\]
\[
= \left[ \prod_{l=1}^{J} \frac{x_l^l}{\pi_l^l} \right] \left( \frac{m!}{x_1!(m - x_1)!} \right) \cdots \frac{(m - x_1)!}{x_2!(m - x_1 - x_2)!} \cdots \frac{(m - x_1 - \cdots - x_{J-1})!}{x_J!(m - x_1 - \cdots - x_J)!}
\]
\[
= \left[ \prod_{l=1}^{J} \frac{x_l^l}{\pi_l^l} \right] \frac{m!}{x_1!x_2! \cdots x_J!}.
\]

\[d. \text{ Properties:} \]

\[i. \quad X_j \sim \text{Bin}(m, \pi_j) \text{ but NOT ind..} \]
ii. $E_{X_j} = m\pi_j$, $\text{Var}_{X_j} = m\pi_j(1 - \pi_j)$.

e. Estimation:

i. m.l.e.s:
   - likelihood $l(\pi; X) = \sum_j X_j \log(\pi_j)$.
   - Consider $\pi_J = 1 - \pi_1 - \cdots - \pi_{J-1}$.
   - Setting $l' = 0$, get $X_j/\pi_j - X_J/\pi_J = 0$.
   - Guess that m.o.m.e.s are solutions, and find that this is indeed the case.
   - Note that if $\hat{\pi}_j \neq 0$ then $l(\pi; X) \to -\infty$ as $\pi_j \to 0 \Rightarrow \hat{\pi}$ is maximizer.

f. Testing:

i. $-2 \log(\Lambda) = \sum_j X_j[\log(X_j/m) - \log(\hat{\pi}_j)] \approx \sum_j(X_j - m\hat{\pi}_j)^2/(m\pi_j)$

ii. Approximation comes from Talor series approximation to
   \begin{align*}
   f(x) &= x[\log(x/m) - \log(\hat{\pi})] \quad \text{about } m\hat{\pi} \\
   f(m\hat{\pi}) &= 0; \quad f'(x) = x[m/x]/m + \log(x/m) - \log(\hat{\pi}) \\
   \text{and } f'(m\hat{\pi}) &= 1; \quad f''(x) = 1/x. \\
   f(x) &\approx (x - m\pi) + (x - m\pi)^2/(2m\pi) \\
   -2 \log(\Lambda) &\approx \sum_j[2(X_j - m\pi_j) + (X_j - m\pi_j)^2/(m\pi_j)] =
\[
\sum_j (X_j - m\pi_j)^2 / (m\pi_j).
\]

- Approximation is Pearson’s $\chi^2$ test

iii. Application: goodness of fit testing