iii. Case 2: $\sigma = \tau$, but common value is unknown.
- Estimate common value by
  $$s_p = \sqrt{\frac{\sum_{j=1}^n (X_j - \bar{X})^2 + \sum_{j=1}^m (Y_j - \bar{Y})^2}{n + m - 2}}$$
- Homework shows that the result is $\sim \chi^2(n + m - 2)$
- Hence relation between degrees of freedom and number of observations is more complicated than before.
- Hence CI is
  $$\bar{Y} - \bar{X} \pm s_p \sqrt{\frac{1}{n + 1/m}}$$

iv. Case 3: $\sigma$ and $\tau$ are unknown.
- Estimate separately using usual formulae.
- $T = \frac{\bar{Y} - \bar{X} - (\nu - \mu)}{\sqrt{\sigma^2/(n - 1) + \tau^2/m}}$ $\sim \chi^2$
  - ? depends on relation between $\sigma$ and $\tau$
    - Best case: $\sigma = \tau$ $\Rightarrow$ DF almost $n + m - 2$
    - Worst case: if $n \geq m$, then $\sigma = 0$ $\Rightarrow$ $T(m - 1)$
    - Usual solution: complicated combination of $\hat{\sigma}$ and $\hat{\tau}$.
  - Hence CI is
    $$\bar{Y} - \bar{X} \pm \sqrt{\frac{\sum_{j=1}^n (X_j - \bar{X})^2/(n - 1) + \sum_{j=1}^m (Y_j - \bar{Y})^2/(m - 1)}/mt_{n+m-2,\alpha/2}}$$

F. 11.4

e. Example: Binomial Distribution: $X \sim \text{Bin}(\theta, m)$.
- i. can we create a pivotal quantity?
  - No.
- ii. For small samples confidence intervals may be calculated exactly. The accompanying figures demonstrate construction of exact confidence intervals for small binomial problems.
  - $\forall \theta \in (0, 1)$
    - $\forall q \in [0, 1]$ calculate $P_\theta [Q \leq q]$ for $Q = X/m$
    - select $\max\{q\}P_\theta [Q \leq q] \geq 1 - \alpha/2$.
    - $\forall q \in [0, 1]$ calculate $P [Q \geq \theta]$.
    - select $\min\{q\}P_\theta [Q \geq q] \geq 1 - \alpha/2$.
  - Confidence interval is the region between extreme endpoints for segments.
- iii. For larger $n$, approximations make things easier:
  - Approximate pivotal quantity:
    $$\sqrt{\frac{Q - \theta}{\theta(1 - \theta)/m}} \sim N(0, 1)$$
    $$P \left[ z_{\alpha/2} \leq \frac{Q - \theta}{\sqrt{\theta(1 - \theta)/m}} \leq z_{1-\alpha/2} \right] = 1 - \alpha,$$
    $$P \left[ -z_{1-\alpha/2} \leq \frac{Q - \theta}{\sqrt{\theta(1 - \theta)/m}} \leq z_{1-\alpha/2} \right] = 1 - \alpha.$$