7. Solution using confidence intervals
   a. Null hypothesis $\theta_0$ vs alternative that $\theta = \theta_1 > \theta_0$.
   b. Construct test
      i. Make a $1 - \alpha$ confidence interval
      • One-sided form $\{L(\text{data}), \infty\}$
      ii. Reject $H_0$ if $\theta_0$ is outside the confidence interval; don’t reject otherwise
      iii. Type I error $\alpha$.
   iv. Do the normal approximation confidence interval method and the normal approximation testing method give the same results?
      • confidence interval for binomial probability contains $\theta$ 95% of the time regardless of the value of $\theta$.
      ▶ Hence standard error has $\theta$ replaced by estimate.
      • When testing, critical value depends only the distri-
        of the test statistic when $\theta$ takes null value.
      ▶ Hence we use null value rather than estimate in the standard error.
      \[ F: 12.4 \]

C. Test construction via likelihood functions.
1. Construct a test using the likelihood function
   a. $L(\theta; X_1, \cdots, X_n)$ gave us a measure of how likely a particular $\theta$ value is.
   b. l.r.t. tells us:
      i. To test $H_0 : \theta = \theta_0$ vs $H_A : \theta = \theta_1$, use as the test statistic $T = L(\theta_1; X_1, \cdots, X_n)/L(\theta_0; X_1, \cdots, X_n)$
      ii. and reject when $T > c$.

Lecture 16

b. For $H_0 : \pi = A$ vs $H_A : \pi = B$,
   \[
   \begin{array}{cccc}
   t & P[\sum_i X_i = t; H_0] & P[\sum_i X_i = t; H_A] & \Lambda \\
   0 & 0.0060 & 0.0001 & 0.0173 \\
   1 & 0.0403 & 0.0016 & 0.0390 \\
   2 & 0.1209 & 0.0106 & 0.0878 \\
   3 & 0.2150 & 0.0425 & 0.1975 \\
   4 & 0.2508 & 0.1115 & 0.4444 \\
   5 & 0.2007 & 0.2007 & 1.0000 \\
   6 & 0.1115 & 0.2508 & 2.2500 \\
   7 & 0.0425 & 0.2150 & 5.0625 \\
   8 & 0.0106 & 0.1209 & 11.3906 \\
   9 & 0.0016 & 0.0403 & 25.6289 \\
   10 & 0.0001 & 0.0060 & 57.6650 \\
   \end{array}
   \]

3. Properties of l.r.t.s of size $\alpha$:
   a. Among all tests that have type I error of size no more than $\alpha$ none have a smaller type II error than this test.
   b. Equivalently, among all tests that have type I error of size no more than $\alpha$ none have greater power, and so this is the most powerful test for these hypotheses.
   c. This result is called the Neyman-Pearson Lemma.
   Proof:
      i. Suppose
      • test “Reject if $X \in A \cup B$” is l.r.t., with size $\alpha$,
      and
      • “Reject if $X \in A \cup C$” is competitor, with size $\leq \alpha$.
      ii. Know $P[X \in A \cup C; H_0] \leq P[X \in \{A \cup C \cup B\}; H_0]$
           $\Rightarrow P[X \in C; H_0] \leq P[X \in A \cup C; H_0]$, $\Rightarrow$ \[
   \int_C f(x; \theta_0) \, dx \leq \int_B f(x; \theta_0) \, dx
   \]
      i. we have simple null and simple alternative hypotheses.

2. Examples:
   a. Consider the car example of before, and test the null hypothesis $H_0 : \pi = \pi_0$ vs the alternative $H_A : \pi = \pi_1$.
      i. The likelihood ratio is
      \[ T = \frac{\prod_{j=1}^n \pi_1^{X_j} (1 - \pi_1)^{m - X_j}}{\prod_{j=1}^n \pi_0^{X_j} (1 - \pi_0)^{m - X_j}} \]
      ii. Simplifying,
      \[ T = \frac{\sum_{j=1}^n X_j (1 - \pi_1)^m}{\sum_{j=1}^n X_j (1 - \pi_0)^m} \]
      \[ = \frac{\pi_1 (1 - \pi_0)}{\pi_0 (1 - \pi_1)} \sum_{j=1}^n X_j \frac{(1 - \pi_1)^m}{(1 - \pi_0)} . \]
      iii. Since
      \[ \frac{\pi_1 (1 - \pi_0)}{\pi_0 (1 - \pi_1)} > 1 \]
      $T$ is large when $\sum_{j=1}^n X_j$ is large and small otherwise.
      ▶ Hence this again gives a test that says: Reject $H_0$ if $\sum_{j=1}^n X_j \geq c$ (or reject if $Q > c$, for a different $c$.)