11. Inference about two normal variances:
   a. \( X_1, \ldots, X_m \sim N(\mu, \sigma^2) \) \( Y_1, \ldots, Y_n \sim N(\nu, \tau^2) \)
      i. \( H_0 : \sigma = \tau \) vs. \( H_A : \sigma \neq \tau \)
      b. \( \hat{\mu} = \bar{X}, \; \hat{\nu} = \bar{Y} \)
      c. \( \hat{\sigma} = \sqrt{\frac{1}{m} \sum_{j=1}^{m} (X_j - \bar{X})^2} \),
         \( \hat{\tau} = \sqrt{\frac{1}{n} \sum_{j=1}^{n} (Y_j - \bar{Y})^2} \), \( \hat{\sigma} = \hat{\tau} = \sqrt{\frac{1}{m+n} \left( \sum_{j=1}^{m} (X_j - \bar{X})^2 + \sum_{j=1}^{n} (Y_j - \bar{Y})^2 \right)} \)
         \( m+n \) = \( \sqrt{\frac{\gamma \sigma^2 + (1 - \gamma) \tau^2}{\hat{\sigma}^2}} \) for \( \gamma = \frac{m}{m+n} \)
   d. Likelihood ratio statistic is
      \[ \Lambda = \frac{\exp \left( -\sum_{j=1}^{m} (X_j - \bar{X})^2 \right) \frac{1}{2 \sigma^2} - \sum_{j=1}^{n} (Y_j - \bar{Y})^2 \frac{1}{2 \tau^2} \hat{\sigma} - m \hat{\tau} - n}{\exp \left( -\frac{m+n}{2} \right) \frac{1}{\hat{\sigma} - m \hat{\tau} - n}} \]
   e. When \( m = n, \Lambda = \left( \frac{1}{\hat{\sigma}} \hat{\tau} + \frac{\hat{\tau}}{\hat{\sigma}} \right)^{-\left(m+n\right)/2} \)
      i. Hence reject \( H_0 \) if \( \hat{\sigma}^2 / \hat{\tau}^2 > c \) or \( \hat{\tau}^2 / \hat{\sigma}^2 > c \)
      ii. \( \hat{\tau}^2 / \hat{\sigma}^2 \sim \chi^2_{m-1, m-1} \)
   f. When \( m \neq n \), not exactly right
      F: 13.8

12. Multinomial Inference
   a. Extension of Binomial:
      i. Draw \( m \) items that are either successes or failures.

   \[ P \left[ X = x; \pi \right] = \sum_{y \text{ yielding } x} P \left[ Y = y; \pi \right] \]

   \[ = \prod_{i=1}^{J} \pi_i^{x_i} \times \text{# of } y \text{ associated with } x \]

   \[ = \prod_{i=1}^{J} \pi_i^{x_i} \times \text{# of ways to get } x_1 \times \]

   \# of ways to get \( x_2 \) from remaining \( m - x_1 \times \ldots \)

   \# of ways to get \( x_J \) from remaining \( m - x_1 - \ldots - x_{J-1} \)

   \[ = \prod_{i=1}^{J} \pi_i^{x_i} \left( \frac{m!}{x_1! (m-x_1)!} \frac{m-x_1}{x_2! (m-x_1-x_2)!} \ldots \right) \]

   \[ \frac{m-x_1-\ldots-x_{J-1}}{x_J! (m-x_1-\ldots-x_{J-1})!} \]

   \[ = \prod_{i=1}^{J} \pi_i^{x_i} \frac{m!}{x_1! x_2! \cdots x_J!} \]

d. Properties:
   i. \( X_j \sim \text{Bin}(m, \pi_j) \) but NOT ind.
   ii. \( E X_j = \pi_j, \text{Var} X_j = \pi_j (1 - \pi_j) \)

e. Estimation:
   i. m.l.e.s:

   - Successes have probability \( \pi \)
   - Failures have probability \( 1 - \pi \)

ii. Record total \# of successes.

b. Multinomial Distribution Definition:
   i. Draw \( m \) items that fall into one of \( J \) groups.
      - Group \( j \) has probability \( \pi_j \)
      - Hence \( \sum_j \pi_j = 1 \)

   ii. Record:
      - Raw data \( Y_1, \ldots, Y_m \), where each \( Y_j \in \{1, \ldots, J\} \)
      - Sufficient reduction \( X_1, \ldots, X_J \) the \#s of successes in each group.
      - Hence \( \sum_j X_j = m \)

c. Multivariate p.m.f.:
   i. for raw data: \( P \left[ Y = y; \pi \right] = \prod_{j=1}^{m} \pi_{Y_j} = \prod_{j=1}^{J} \pi_j^{X_j} \)
   ii. for reduced data:

   \[ l(\pi; X) = \sum_j X_j \log(\pi_j) \]

   - Consider \( \pi_j = 1 - \pi_1 - \cdots - \pi_{J-1} \)
   - Setting \( x = 0 \), get \( X_j / \pi_j - X_j / \pi_j = 0 \)
   - Guess that m.o.m.e solutions are solutions, and find that this is indeed the case.
   - Note that if \( \hat{\pi}_j \neq 0 \) then \( l(\pi; X) \to -\infty \) as \( \pi_j \to 0 \Rightarrow \pi \) is maximizer.

f. Testing:
   i. \( -2 \log(A) = \sum_j X_j \left( \log(X_j/m) - \log(\hat{\pi}_j) \right) \approx \sum_j (X_j - m \hat{\pi}_j)^2 / (m \hat{\pi}_j) \)
   ii. Approximation comes from Talor series approximation to \( f(x) = x \log(x/m) - \log(\hat{\pi}) \) about \( m \hat{\pi} \)
      \[ f(m \hat{\pi}) = 0; \; f'(x) = x [\log(x/m) + \log(\hat{\pi}) - \log(\hat{\pi})] \] and \( f''(m \hat{\pi}) = 1; \; f''(x) = 1/x \.
      \[ f(x) \approx (x - m \hat{\pi}) + (x - m \hat{\pi})^2 / (2m \hat{\pi}) \]
   - \( -2 \log(A) \approx \sum_j [2(X_j - m \hat{\pi}_j) + (X_j - m \hat{\pi}_j)^2 / (m \hat{\pi}_j)] = \sum_j (X_j - m \hat{\pi}_j)^2 / (m \hat{\pi}_j) \)
   - Approximation is Pearson’s \( \chi^2 \) test
   iii. Application: goodness of fit testing