VIII. Models and Graphs

A. The rule: Never put a term in the model without lower order terms
   1. Ex. all models we examine effectively have an intercept
   2. Ex. a model with $X \times Y$ interactions must also have $X$ and $Y$ main effects
   3. Ex. a model with $X \times Y \times Z$ interactions must also have $X$, $Y$, $Z$ main effects, $X \times Y$, $X \times Z$, $Y \times Z$ interactions.

B. The Picture:
   1. Make dots (vertices) corresponding to main effects in the model
   2. Join dots by lines (edges) if associated two-way interaction is in the model.
      a. Edges are undirected.
   3. Examples for model with four variables $W$, $X$, $Y$, $Z$ are in Figure 11/.

C. Graphical models:
   1. Sets of vertices connected all to each other are called cliques.
   2. A model is called graphical if higher–order interactions are present to connect cliques, and are absent otherwise.
D. Conditional Independence:

1. A path is a sequence of edges leading from one vertex to another.
2. If all paths leading from one vertex to another vertex run through
a third vertex, then the associated variables for the first two vertices are independent conditional on the third vertex.

3. Works for sets of variables rather than just variables if model is graphical.

4. Example: \[ \begin{array}{c}
W \longrightarrow X \longrightarrow Y 
\end{array} \]

a. Hence model is \[ \log(\lambda_{wxy}) = \alpha_w W + \alpha_x X + \alpha_y Y + \alpha_{wx} W X + \alpha_{xy} XY \]

b. In full multinomial model, \[ P[W = w, X = x, Y = y] = \frac{\exp(\alpha_w W + \alpha_x X + \alpha_y Y + \alpha_{wx} W X + \alpha_{xy} XY)}{C} \]

\[ C = \sum_{s,t,y} \exp(\alpha_s W_s + \alpha_t X_t + \alpha_y Y_y + \alpha_{st} W X + \alpha_{ty} XY). \]

c. In full multinomial model, \[ P[X = x] = \exp(\alpha_x X) [\sum_s \exp(\alpha_s W + \alpha_{sx} W X)] [\sum_u \exp(\alpha_u Y + \alpha_{yu} XY)] / C. \]

d. In full multinomial model, \[ P[W = w, Y = y | X = x] = \frac{\exp(\alpha_w W + \alpha_{wx} W X)}{\sum_s \exp(\alpha_s W + \alpha_{sx} W X)} \frac{\exp(\alpha_y Y + \alpha_{xy} XY)}{\sum_t \exp(\alpha_t Y + \alpha_{tx} XY)}. \]

e. \[ W \perp Y | X \] (i.e., \( W \) is independent of \( Y \) conditional on \( X \)).

E. Collapsibility:

1. When can we ignore a variable from a model?

a. Want parameters to have the same interpretation, and have the same estimates

b. Depends on what parameters you are interested in.
2. Conditions:
   a. If it’s independent of all other variables
      i. ie, no interactions
      ii. Not very informative
   b. If variable is related to only one of the variables, you can collapse over it when doing inference on interactions between these variables

3. Ex., for the model $W \quad X \quad Y$
   a. ie., $\log(P[W = w, X = x, Y = y]) = \mu + \alpha_W^W + \alpha_X^X + \alpha_Y^Y + \alpha_{WX}^W + \alpha_{XY}^X$
   b. You can collapse over $Y$ for inference in $\alpha_{WX}^W$
   c. You can collapse over $W$ for inference in $\alpha_{XY}^X$

4. Example in symbols:
   a. In full multinomial model, $P [X = x, Y = y] = \exp(\alpha_X^X + \alpha_Y^Y + \alpha_{XY}^X) [\sum_s \exp(\alpha_s^W + \alpha_{sX}^W)]/C = \exp(\beta_X^X + \alpha_Y^Y + \alpha_{XY}^X)/C$
      for $\beta_X^X = \alpha_X^X + \log([\sum_s \exp(\alpha_s^W + \alpha_{sX}^W)])$
   b. Definition of main effect for $X$ changes, but interaction between $X$ and $Y$ stays the same.

5. MLEs exactly the same only if only a single variable is attached to
A: 7.2, 7.3.4

F. Alternative to interaction is one-degree-of-freedom parameterization $\alpha^{XY} = \beta u_x v_y$