1. Calculate the asymptotic relative efficiency for the Wilcoxon signed rank statistic relative to the one-sample t-test (which you should approximate using the one-sample z-test).

The T-test is approximated as $T_1 = \sum_{j=1}^{n} X_j / n$. Then $\mu_1(\theta) = \theta$, and $\mu_1'(0) = 1$. Also, $\sigma_1(0) = \sqrt{n \text{Var}[T_1]} = \sqrt{\text{Var}[X_j]}$. The Wilcoxon test might be expressed as the average of the indicators of positive Walsh averages. That is, let $T_2 = \sum_i \sum_{j \geq i} I(X_i + X_j \geq 0)/(n(n+1)/2)$. For large sample sizes $n$, averages of observations with themselves constitute a vanishing portion of the collection of all Walsh averages, and so $\mu_2(\theta) = P[X_1 + X_2 \leq 2\theta] + O(1/n)$, where $X_j$ are the observations. The variance of the sum of indicators is $n(n+1)(2n+1)/24$, and so the variance of the average is $(n(n+1)(2n+1)/24)(n(n+1)/2)^{-2} \approx 1/(3n)$. So, $\sigma_2(0) = \sqrt{1/3}$.

Do this for observations from

a. the distribution uniform on the interval $[\theta - 1/2, \theta + 1/2]$.

Here, $\sigma_1(0) = \sqrt{1/12}$, and the efficacy of the T-test is $e_1 = 2\sqrt{3}$. Also

$$\mu_2(\theta) = \int_{1/2}^{1} \int_{1/2}^{1/2} dx_1 dx_2 = \int_{1}^{1} (1/2 - x_2) dx_2 = -2\theta^2 + 2\theta + 1/2.$$ 

So $\mu_2'(0) = 2 - 4\theta$, and $\mu_2'(0) = 2$. So the efficacy for the Signed Rank Test is $e_2 = 2/\sqrt{1/3} = 2\sqrt{3} = 3.41$. The asymptotic relative efficiency is 1.

To verify via simulation, simulate a large number of large uniform data sets, offset by a small amount, and calculate powers for signed rank and z-tests:

```r
out<-array(NA,c(10000,2))
for(j in seq(dim(out)[1])){
  x<-runif(500)-1/2+.04;
  out[j,]<-c(t.test(x)$p.value,wilcox.test(x)$p.value)<0.025
}
apply(out,2,mean)
```

to approximate powers for equal sample sizes as 0.8074 and 0.7745, which are very similar.

b. the logistic distribution, symmetric about $\theta$, with variance $\pi^2/3$ and density $\exp(x-\theta)/(1+\exp(x-\theta))^2$.

As above, the T statistic has $\mu_1'(0) = 1$, and $\sigma_1(0) = \pi/\sqrt{3}$, with efficacy $e_1 = \sqrt{3}/\pi$. The Wilcoxon statistic has mean $\mu_2(\theta) = P[X_1 + X_2 < 2\theta]$. From (logisticmeanderiv)

$$\frac{d}{d\theta} P[X_1 + X_2 < \xi]\bigg|_{\xi=0} = \frac{1}{6},$$

and so $\frac{d}{d\theta} P[X_1 + X_2 < 2\xi]\bigg|_{\xi=0} = \frac{1}{3}$. Hence the efficacy is $\sqrt{1/3} \approx \sqrt{1/3}$, and the asymptotic relative efficiency is $(\sqrt{1/3}/(\sqrt{3}/\pi))^2 = (\pi/3)^2 = 1.10$, indicating that the Wilcoxon statistic is 10% more efficient than the T-test for the logistic distribution.

c. and the standard normal distribution, with variance 1 and expectation $\theta$.
\[\sigma_1(0) = \sqrt{\text{Var}(X_1)} = 1, \text{ and the efficacy is } e_1 = 1. \text{ So } \mu_2'(\theta) = \exp(-\theta^2)(2\pi)^{-1/2}2^{1/2}, \text{ and } \\
\mu_2'(0) = 1/\sqrt{\pi}. \text{ So the efficacy is } (1/\sqrt{\pi})/\sqrt{1/3} = \sqrt{3/\pi} = 0.977. \text{ The asymptotic relative efficiency is } 0.977^2.\]

2. The data set
http://ftp.uni-bayreuth.de/math/statlib/datasets/federalistpapers.txt
gives data from an analysis of a series of documents. The first column gives document number, the second gives the name of a text file, the third gives a group to which the text is assigned, the fourth represents a measure of the use of first person in the text, the fifth presents a measure of inner thinking, the sixth presents a measure of positivity, and the seventh presents a measure of negativity. There are other columns that you can ignore. (The version on line, above, has odd line breaks. A fixed version can be found at stat.rutgers.edu/home/kolassa/960-555/federalistpapers.txt).

a. Test the null hypothesis that negativity is equally distributed across the groups using a Kruskal-Wallis test.

The following R commands will do this calculation:

```r
fed<-as.data.frame(scan("federalistpapers.txt", 
  what=list(nn=0,fn="",grp=0,firstp=0,inner=0,pos=0,neg=0),
  skip=7,flush=T))
kruskal.test(neg~as.factor(grp),data=fed)
```

The \( p \)-value is 0.03431. Reject the null hypothesis of identical dispersion of inner thinking between groups.

b. Test at \( \alpha = 0.05 \) the pairwise comparisons for negativity between groups using the Bonferroni adjustment, and repeat for Tukey’s HSD.

The following R commands will do this calculation:

```r
fed<-as.data.frame(scan("federalistpapers.txt", 
  what=list(nn=0,fn="",grp=0,firstp=0,inner=0,pos=0,neg=0),
  skip=7,flush=T))
pairwise.wilcox.test(fed$neg,fed$grp,method="bonferroni")
library("MultNonParam")
tukey.kruskal.test(fed$neg,fed$grp)
```

Both adjustments reject only the 1 vs. 3 comparison.

3. The data set
http://ftp.uni-bayreuth.de/math/statlib/datasets/Plasma_Retinol
gives data relating various quantities, including smoking status (1 never, 2 former, 3 current) in column 3 and beta plasma in column 13. Perform a nonparametric test to investigate an ordered effect of smoking status on beta plasma.

The following R commands will do this calculation:
smoke <- as.data.frame(scan("Plasma_Retinol", what=list(AGE=0, SEX=0, SMOKSTAT=0, QUETELET=0, VITUSE=0, CALORIES=0, FAT=0, FIBER=0, ALCOHOL=0, CHOLESTEROL=0, BETADIET=0, RETDIET=0, BETAPLASMA=0, RETPLASMA=0), skip=30, nmax=315))
library("clinfun")
jonckheere.test(smoke$BETAPLASMA, smoke$SMOKSTAT)

The two-sided p-value is 0.001546.

4. Demonstration that the two-sample t-statistic for Gaussian data follows the expected distribution uses the fact that the mean differences are independent of the pooled standard deviation estimate, and that the pooled variance estimate times degrees of freedom has a \( \chi^2 \) distribution with the expected number of degrees of freedom. Using simulation, for tests arising from samples of size \( M_1 = M_2 = 10 \), evaluate both of these assumptions for data coming from a Laplace distribution, and from a Cauchy distribution.

The following R code will do this for Laplace:

```r
library(PictexPlot)
library(VGAM)
checkindependence <- function(distn="Laplace", nx=10, ny=10, nsamp=10000){
  rgenx <- paste("r", tolower(distn), "("nx", ")", sep="")
  rgeny <- paste("r", tolower(distn), "("ny", ")", sep="")
  out <- array(NA, c(nsamp, 2))
  for (i in seq(dim(out)[1])){
    x <- eval(parse(text=rgenx))
    y <- eval(parse(text=rgeny))
    out[i,1] <- mean(y) - mean(x)
    out[i,2] <- (sum((y-mean(y))^2) + sum((x-mean(x))^2)) / (nx+ny-2)
  }
  meanq <- apply(outer(out[,1], quantile(out[,1], (1:9)/10), "-" > 0, 1, "sum") + 1
  ssub <- paste(distn, "distribution, sample sizes"nx, "and"ny, 
    ". Outliers are not represented." , sep=" ")
  boxplot(split(out[,2], meanq), outline=FALSE,
    xlab="Mean difference decile", ylab="Standard Deviation",
    main="Pooled Standard Deviation by mean difference decile", sub=ssub)
  zz <- qqplot(x=outer(out[,2] * (nx+ny-2), y=rchisq(20000, df=nx+ny-2),
    xlab="Chi-square quantiles", ylab="Observed Quantiles",
    main="Quantile plot for sample variance vs. chi-square", sub=ssub)
  return(zz)
}
a <- checkindependence()
b <- checkindependence(distn="Cauchy")
```
Figures 1 and 2 demonstrate boxplots of standard deviation by decile of mean difference. Both show that standard deviation depends on mean difference.

![Fig. 1: Pooled Standard Deviation by mean difference decile](image1)

**Mean difference decile**
Laplace distribution, sample sizes 10 and 10. Outliers are not represented.

![Fig. 2: Pooled Standard Deviation by mean difference decile](image2)

**Mean difference decile**
Cauchy distribution, sample sizes 10 and 10. Outliers, and most extreme mean difference deciles, are not represented.

Figures 3 and 4 are quantile plots for the $\chi^2$ distribution with the appropriate degrees of freedom. If the normal-theory approximation held, these would be a straight line; this clearly fails here.
Fig. 3: Quantile plot for Pooled Standard Deviations for Laplace Variables

Laplace distribution, sample sizes 10 and 10.

Fig. 4: Quantile plot for Pooled Standard Deviations for Cauchy Variables

Cauchy distribution, sample sizes 10 and 10. Six largest observation-quantile pairs are removed.