X. Bivariate semi-parametric methods:
A. Recall Spearman’s Rank Correlation $r_s$ as correlation of ranks of $X$’s and $Y$’s. Motivation was that under the alternative large $X$’s would be associated with large $Y$’s, or with small $Y$’s.
B. Assume a linear model $Y_j = \beta_1 + \beta_2 X_i + e_i$, $e_i$ has some distribution with median 0 (making $\beta_1$ identifiable).
C. Two objectives:
1. We already know how to test $\beta_2 = 0$;
2. Now we estimate $\beta_2$.
D. Get confidence interval by inverting test of $\tau = \tau(\beta_2) = 0$, based on $r_T$.
1. Could also do this using Spearman.
2. Could also do this using Pearson correlation under permutation distribution.
3. Could also do this using permutation distribution for $\sum_{j=1}^n (X_j - \bar{X})R_j$ for $R_j$ the rank of the residuals.
   a. or use transformation of ranks.
E. Steps to invert the test using correlation between $X_i$ and $Y_i - \beta_2 X_i$, as a function of $\beta_2$
1. Calculate $q_{\alpha/2}$ and $q_{1-\alpha/2}$: lower and upper 2.5% points of distribution of $r$, $r_S$, $r_T$.
   a. Since we know null mean and variance, this could be approximated using normal distribution.
2. Solve $r(\beta_2) = q_{\alpha/2}$, $r(\beta_2) = q_{1-\alpha/2}$.
F. Both parts are tricky for Pearson correlation (with permutation distribution).

B. Simple case: one sample.
1. Model: $X_i \sim F_X(x)$
2. Normal case, known variance $\Sigma$:
   a. Let $\bar{X} = \sum_{i=1}^n X_i/n$
3. Null hypothesis: marginal medians take on prespecified values
   a. WOLOG, value is zero.
   b. That is, $F_X(\infty, \ldots, \infty, 0, \infty, \ldots, \infty) = 1/2$ for each possible slot for zero.
   c. Less sloppily, let
      i. $z_j$ be the vector with $j$ components,
      ii. with all components $\infty$ except for that in component $j$
   iii. $H_0: F_X(z_j) = 1/2$
4. Normal case:
   a. Let $\bar{X} = \sum_{i=1}^n X_i/n$
   b. $\Sigma$ known: $\bar{X}^\top \Sigma^{-1} \bar{X} \sim \chi_j^2$, if $\Sigma$ nonsingular.
   c. $\Sigma$ unknown: estimate using usual sum of squares: $\bar{X}^\top \Sigma^{-1} \bar{X} \sim F_{l,j}$, if $\Sigma$ nonsingular.
   d. $\Sigma$ unknown, but sum of squares estimator not appropriate: $\bar{X}^\top \Sigma^{-1} \bar{X} \sim \chi_j^2$ approximately.
   e. Techniques require multivariate normality, which is stronger than marginal normality.
5. Nonparametric Solution: combine univariate rank statistics for marginal test $T$:
   a. Sign test, or signed rank test assuming symmetry (often in the context of paired data).
   i. Solution depends on approximate normality of $T$
   ii. We’ve claimed before that the components of $T$ are separately approximately normal, by reference to a CLT
   iii. Similar arguments work for the vector as a whole.
b. Difficulties:
   i. Separate tests are generally dependent, and dependence structure depends on distribution of raw observations.
   • We will have to estimate this.
   ii. Null hypothesis dependent on the coordinate system for variables, but analysis does not.
   • Ex. If $(X_i, Y_i) \sim N(v, \Sigma)$ with $\Sigma$ known, and $H_0: v = 0$, then the canonical test is $(X_i, Y_i) \Sigma^{-1} (X_i, Y_i)^\top$, and it is unchanged if we base test on $(U_i, V_i)$ for $U_i = X_i + Y_i$, and $V_i = X_i - Y_i$.
c. Solution for sign test:
   i. Let $T_j = \sum_i s(X_j)$ for $s(u) = \begin{cases} 1 & \text{if } u > 0 \\ -1 & \text{if } u < 0 \end{cases}$.
   ii. Then under $H_0$, $T_j/\sqrt{n} \approx N(0,1)$.
   iii. Estimate $\text{Cov}(s(X_{ij}), s(X_{ij}')) = E[s(X_{ij})s(X_{ij}')]$ by $\hat{\sigma}_{jj'} = \sum_i s(X_{ij})s(X_{ij}')/n$.
   iv. So test using $T^\top \begin{pmatrix} 1 & \hat{\sigma}_{12} & \cdots \\ \hat{\sigma}_{21} & 1 & \cdots \\ \vdots & \vdots & \ddots & \ddots \\ \vdots & \vdots & \ddots & \ddots \end{pmatrix}^{-1} T/n \sim \chi_j^2$ under $H_0$.
d. Solution for Wilcoxon signed rank test is similar.
6. Permutation Solution:
a. Select an existing test statistic
   i. Hotelling
   ii. Rank-based
   iii. etc.

b. Compare value against permutation distribution
   i. Randomly reassign signs of observation vectors as a whole.

C. Confidence Regions for parameter vector $\mu$
   1. Introduce shift parameter to move data to conventional null hypothesis; for ex.,
      a. One-sample: $X \mapsto X - 1_n \otimes \mu$
         i. $1_n$ is vector of ones of length $n$
         ii. $\otimes$ is outer product, so $1 \otimes \mu_n$ is the matrix with entry $\mu_j$ in column $j$ for all rows.