On the Convolution of Cauchy Distributions
Author(s): Meyer Dwass
Published by: Taylor & Francis, Ltd. on behalf of the Mathematical Association of America
Stable URL: https://www.jstor.org/stable/2322198
Accessed: 08-02-2019 20:22 UTC

JSTOR is a not-for-profit service that helps scholars, researchers, and students discover, use, and build upon a wide range of content in a trusted digital archive. We use information technology and tools to increase productivity and facilitate new forms of scholarship. For more information about JSTOR, please contact support@jstor.org.

Your use of the JSTOR archive indicates your acceptance of the Terms & Conditions of Use, available at https://about.jstor.org/terms
bijective proof of (1), utilizing their Involution Principle. We are going to give another bijective
proof which does not require any iterations and is very simple. Indeed,

\[ \phi: U Par(n - a(j)) \leftrightarrow U Par(n - a(j)), \]

given below does the job.

Let \( (\lambda) = (\lambda(1), \ldots, \lambda(t)) \in Par(n - a(j)) \). Then define \( \phi \) by

\[
\phi((\lambda)) = \begin{cases} 
(t + 3j - 1, \lambda(1) - 1, \ldots, \lambda(t) - 1) \in Par(n - a(j - 1)), & \text{if } t + 3j \geq \lambda(1), \\
(\lambda(2) + 1, \ldots, \lambda(t) + 1, 1, 1, \ldots, 1) \in Par(n - a(j + 1)), & \text{if } t + 3j < \lambda(1), 
\end{cases}
\]

where there are \( \lambda(1) - 3j - t - 1 \) 's at the end.

Note that applying \( \phi \) twice yields the identity mapping, thus \( \phi = \phi^{-1} \) and \( \phi \) is invertible.

Example. \( n = 21 \).

\[ \phi(5, 5, 4, 3, 2) = (7, 4, 4, 3, 2, 1). \]

Here \( (5, 5, 4, 3, 2, ) \in Par(19) = Par(n - a(1)) \) so \( j = 1 \). The number of parts \( t \), is 5 and we have \( t + 3j \geq \lambda(1) \), since \( 5 + 3 \geq 5 \). Now consider \( \phi(7, 4, 4, 3, 2, 1) \); here \( j = 0 \), \( t = 6 \), \( \lambda(1) = 7 \) and \( 6 + 0 < 7 \). Also \( \lambda(1) - 3j - t - 1 = 7 - 0 - 6 - 1 = 0 \) so we do not add any 1's at the end and \( \phi(7, 4, 4, 3, 2, 1) = (5, 5, 4, 3, 2) \).

References


---

THE TEACHING OF MATHEMATICS

EDITED BY MARY R. WARDROP AND ROBERT F. WARDROP

Material for this department should be sent to Professor Robert F. Wardrop, Department of Mathematics, Central Michigan University, Mount Pleasant, MI 48859.

ON THE CONVOLUTION OF CAUCHY DISTRIBUTIONS

MEYER DWASS

Department of Mathematics, Northwestern University, Evanston, IL 60201

The characteristic function of a probability distribution is usually too advanced a topic for a first undergraduate course in mathematical statistics and the more limited moment generating function is often used instead. In teaching the distribution of sums of independent random variables such as normal, gamma, or uniform, I supplement the use of the moment generating function with the convolution formula,

\[ f \ast g(u) = \int_{-\infty}^{\infty} f(x) g(u - x) \, dx. \]

For sums of independent Cauchy random variables the moment generating function does not apply and the use of the convolution formula is difficult. Undoubtedly, it is generally understood that if \( f \) and \( g \) are Cauchy densities, a partial fraction decomposition of the integrand in (1) should lead to an explicit evaluation of the convolution integral but I do not find the details worked out anywhere. (See comment in Feller, [1], p. 51.) The purpose of this note is to outline...
these details as I have presented them in the classroom. The algebra is elementary and well within
the scope of an undergraduate course.

The calculation is based on the following relationship:

**Lemma.** For $a > 0$, let

$$f(x,a) = \frac{1}{\pi a \left[ 1 + (x/a)^2 \right]}, \quad -\infty < x < \infty.$$  

Then

$$f(x,a)f(u-x,b) = \frac{[a + b - ux/a] f(x,a) + [a + b - u(u-x)/b] f(u-x,b)}{2\pi \left[ (a + b)^2 + u^2 \right]}$$

$$+ \frac{[-a + b + ux/a] f(x,a) + [-a + b - u(u-x)/b] f(u-x,b)}{2\pi \left[ (a - b)^2 + u^2 \right]}.$$  

From the lemma it follows immediately that

$$(2) \quad \int_{-\infty}^{\infty} f(x,a)f(u-x,b) \, dx = f(u,a+b),$$  

since the integral in (2) exists and

$$\int_{-\infty}^{\infty} f(x,a)f(u-x,b) \, dx = \lim_{T \to \infty} \int_{-T}^{T} f(x,a)f(u-x,b) \, dx$$  

and

$$\lim_{T \to \infty} \int_{-T}^{T} xf(x,a) \, dx = \lim_{T \to \infty} \int_{-T}^{T} (u-x)f(u-x,b) \, dx = 0.$$  

**Proof of Lemma.** For $a > 0$, define

$$h(x,a) = \frac{1}{2\pi a (1 + ix/a)},$$

$$h(x,-a) = \frac{1}{2\pi a (1 - ix/a)},$$

where $i = \sqrt{-1}$.

The following steps follow by routine algebra and the details are left to the reader:

(a) $f(x,a) = h(x,a) + h(x,-a),$

(b) $f(x,a)f(u-x,b) = h(x,a)h(u-x,b) + h(x,a)h(u-x,-b) + h(x,-a)h(u-x,b) + h(x,-a)h(u-x,-b),$

(c) $h(x,a)h(u-x,b) = \frac{h(x,a) + h(u-x,b)}{2\pi (a + b + iu)},$

$$h(x,a)h(u-x,-b) = -\frac{h(x,a) + h(u-x,-b)}{2\pi (a - b + iu)},$$

$$h(x,-a)h(u-x,b) = -\frac{h(x,-a) + h(u-x,b)}{2\pi (a - b - iu)},$$

$$h(x,-a)h(u-x,-b) = \frac{h(x,-a) + h(u-x,-b)}{2\pi (a + b - iu)},$$
(d) \( h(x, a)h(u - x, b) + h(x, -a)h(u - x, b) = \)
\[
\frac{\left[ a + b - ux/a \right] f(x, a) + \left[ a + b - u(u - x)/b \right] f(u - x, b)}{2\pi \left( (a + b)^2 + u^2 \right)}.
\]
\[24 \left( a + b \right)^2 + u^2\]

\( h(x, a)h(u - x, -b) + h(x, -a)h(u - x, b) = \)
\[
\frac{\left[ -a + b + ux/a \right] f(x, a) + \left[ a - b + u(u - x)/b \right] f(u - x, b)}{2\pi \left( (a - b)^2 + u^2 \right)}.
\]
\[24(a - b)^2 + u^2\]

The lemma now follows by combining the last two equations.

Reference


PROBLEMS AND SOLUTIONS

EDITED BY G. L. ALEXANDERSON, DAVID BORWEIN (ADVANCED PROBLEMS), H. M. W. EDGAR (ELEMENTARY PROBLEMS), AND D. H. MUGLER


Send all proposed problems, typed and in duplicate if possible, to Professor D. H. Mugler, Department of Mathematics, University of Santa Clara, Santa Clara, CA 95053. Please include solutions, relevant references, etc.

An asterisk (*) indicates that neither the proposer nor the editors supplied a solution.

Solutions should be sent to the address given at the head of each problem set.

A publishable solution must, above all, be correct. Given correctness, elegance and conciseness are preferred.

The answer to the problem should appear right at the beginning. If your method yields a more general result, so much the better. If you discover that a MONTHLY problem has already been solved in the literature, you should of course tell the editors; include a copy of the solution if you can.

ELEMENTARY PROBLEMS

Solutions of these Elementary Problems should be mailed in duplicate to Professor D. H. Mugler, Department of Mathematics, University of Santa Clara, Santa Clara, CA 95053, by May 31, 1985. Please place the solver’s name and mailing address on each (double-spaced) sheet. Include a self-addressed card or label (for acknowledgement).

E 3069. Proposed by Zhang Zaiming, Yuxi Teachers’ College, Yunnan, China.

The \( m \times m \) determinant \( I = \| a_{rs} \| \) has \( a_{rs} = \int_0^1 x^{r-1}[F(x)]^{m-r+1} \, dx \), where \( F \) is nondecreasing on \([0,1]\). Prove that \( I \geq 0 \).

E 3070. Proposed by Gérard Letac, Université Paul Sabatier, Toulouse, France.

Let \( a \) and \( (t_n)_{n \geq 1} \) be strictly positive numbers and let the sequence \( (t_n)_{n \geq 1} \) be bounded. Prove