Exam 2

1. Suppose that $X \sim \text{Bin}(\pi, m)$ and $Y \sim \text{Bin}(\theta, n)$, and are independent.
   
a. Describe a test of the null hypothesis $\theta = \pi$ vs. the alternative $\theta \neq \pi$ that uses the normal approximation to the binomial distribution. Include a formula for the test statistic, a description of what values of the test statistic will indicate rejection of the null hypothesis, and a description of the test statistic’s null distribution.
   
   The statistic is
   
   $$Z = \frac{Y/n - X/m}{\sqrt{\frac{X + Y}{m + n} \left(1 - \frac{X + Y}{m + n}\right)(1/m + 1/n)}}.$$ 

   Reject the null hypothesis if $|Z| > z_{\alpha/2}$, where $\Phi(z_{\alpha/2}) = \alpha/2$. Here $\Phi$ is the standard normal cumulative distribution function. Here $Z$ is approximately standard normal.

   b. You note that the true distribution of any reasonable test statistic for this problem depends on the common value of $\theta$ and $\pi$, even under $H_0$. Describe a method for testing $\theta = \pi$ that removes this dependence.

   Condition on the sum $X + Y$. Then, under the null hypothesis, $X|X + Y$ is hypergeometric. Reject if $X > c_u$ or if $X < c_l$. Here $c_1$ and $c_u$ are chosen so that $P[X > c_u] \leq \alpha/2$ and $P[X < c_l] \leq \alpha/2$, where the probabilities are from the hypergeometric distribution.

   Total for this question: 7.

2. An investigator gathers data generated according to a certain probability model, with an unknown parameter $\theta$. A continuous random variable $T$ is sufficient for $\theta$. The investigator wishes to test a null hypothesis of the form $\theta = \theta_0$, against the alternative $\theta \neq \theta_0$.

   a. The accompanying figure shows the generalized likelihood ratio statistic $\Lambda(T)$ as a function of $T$. A value of $T = 3$ is observed. Indicate on the picture the region that one must integrate the density over to calculate the $p$-value.

   b. Again assume that the investigator observes the value $T = 3$. The investigator knows the numeric value of $\Lambda$. How might the investigator decide whether to reject the null hypothesis on this basis, without any substantial additional calculation?

   The investigator can use the fact that $-2\ln(\Lambda)$ is distributed $\chi^2_1$, and so reject the null hypothesis if and only if $-2\ln(\Lambda) \geq 1.96^2$.

   c. With a considerable amount of difficulty, the investigator is able to calculate the exact distribution function for $\Lambda$ under both the null hypothesis and under an interesting member of the alternative hypothesis. What is the critical value for a test of size $\alpha = .2$?

   We reject the null hypothesis for small values of $\Lambda$. Find the point on the null curve where .2 probability lies below it. From the picture, it looks like .8.
d. Approximate the power of the test described in (c) above.
The power is the probability on the alternative curve, at the critical value. This is somewhere just above .5.

Total for this question: 10.

3. Suppose that random variables \( X_1, \ldots, X_n \) are independent and that each has the same probability mass function
\[
p_X(x; \theta) = x \exp(\theta x - \theta)(1 - \exp(\theta))^2,
\]
for \( \theta < 0 \) and \( x \in \{1, 2, 3, \ldots\} \). You may use without proof the fact that the maximum likelihood estimator for \( \theta \) is \( \log((\bar{X} - 1)/(\bar{X} + 1)) \).

a. Calculate the statistic for testing the null hypothesis that \( \theta = -1/2 \) vs. the alternative hypothesis that \( \theta = -1 \).

The likelihood is \( \prod_{i=1}^n X_i \exp(\theta X_i - \theta)(1 - \exp(\theta))^2 \), and the Neyman-Pearson statistic is
\[
\frac{\prod_{i=1}^n X_i \exp((-1/2)X_i + 1/2)(1 - \exp(-1/2))^2}{\prod_{i=1}^n X_i \exp(-X_i + 1)(1 - \exp(-1))^2} = \frac{\exp(\theta \sum_{i=1}^n X_i + n/2)(1 - \exp(-1/2))^{2n}}{\exp(\theta \sum_{i=1}^n X_i + n)(1 - \exp(-1))^{2n}}
\]
\[
= \exp((-n\bar{X}/2)/\exp(-n\bar{X})C = \exp(n\bar{X}/2)C,
\]
for \( C \) a constant. We want to reject \( H_0 \) when this is small. Hence the test is of form “Reject \( H_0 \) if \( \bar{X} \leq c \)” for some constant \( c \).

b. Suppose that you observe \( \bar{X} = 1 \). What is the \( p \) value for the test in part (a)? (Hint: If \( \bar{X} = 1 \), what are the possible values for \( X_j \)?)

Note that 1 is the smallest value that \( \bar{X} \) can take; hence the \( p \)-value is
\[
P[\bar{X} = 1] = P[X_j = 1 \forall j] = [1 \exp(-1/2 - 1/2)(1 - \exp(-1/2))^2]^n = (1 - \exp(-1/2))^{2n} = 0.152^n.
\]

c. How would the answer to part (a) change if the alternative hypothesis was \( \theta = -3/4 \)?

The Neyman Person statistic is now \( \exp(-\bar{X}/2)/\exp(-3\bar{X}/4) = \exp(\bar{X}/4) \), times a different constant. Again reject when this is small. Again, “Reject \( H_0 \) if \( \bar{X} \leq c \)” for some constant \( c \), where \( c \) depends only on the null. No change.

d. How would the answer to part (a) change if the alternative hypothesis was \( \theta = -1/4 \)?

The critical region would change to “Reject \( H_0 \) if \( \bar{X} \geq c \)” for some constant \( c \).

e. What useful property does your test have?

It is most powerful.

Total for this question: 13.
4. Suppose that $X$ has a Poisson distribution with mass function
$$p_X(x; \theta) = \exp(-\theta)\theta^x/x!,$$
for $\theta > 0$ and $x \in \{0, 1, 2, 3, \ldots\}$. Suppose that $\theta$ has the Gamma prior density
$$\pi(\theta) = \exp(-\theta)\theta^{k-1}/\Gamma(k).$$

a. Calculate the posterior density for $\theta$. Include the constant to make it integrate to 1.

The posterior is
$$\frac{\exp(-\theta)\theta^x/x!\exp(-\theta)\theta^{k-1}/\Gamma(j)}{\int_0^\infty \exp(-\theta)\theta^x/x!\exp(-\theta)\theta^{k-1}/\Gamma(j) \, d\theta} = \frac{\exp(-2\theta)2^{x+k}\theta^{k+x-1}}{\Gamma(k+x)}.$$

b. The likelihood-prior pair has a convenient property. What is the name of this property?

Conjugate

c. The HPD region $(L, U)$ with posterior probability $1 - \alpha$ can be shown to be the solution to a system of two equations. What are these equations? You need not solve them.

$$\frac{\exp(-2L)2^{k+x}L^{k+x-1}}{\Gamma(k+x)} = \frac{\exp(-2U)2^{k+x}U^{k+x-1}}{\Gamma(k+x)} \leftrightarrow \exp(-2L)L^{k+x-1} = \exp(-2U)U^{k+x-1} \quad \text{and}$$

$$\int_L^U \frac{\exp(-2\theta)2^{k+x}\theta^{k+x-1}}{\Gamma(k+x)} \, d\theta = 1 - \alpha.$$

d. Calculate the posterior mode for $\theta$.

The derivative of the log of the posterior is $\frac{k-2\theta+x-1}{\theta}$. Setting this to zero gives a putative mode of $(x + k - 1)/2$. The second derivative of the log posterior is $-(x + k - 1)\theta^{-2} < 0$, showing that the putative mode is actually the true unique mode.