1. **Question 8.4.**

   a. By the equation $\text{MSE} \left[ \hat{\theta} \right] = \text{Bias} \left[ \hat{\theta} \right]^2 + \text{Var} \left[ \hat{\theta} \right]$, for an estimator with $\text{Bias} \left[ \hat{\theta} \right] = 0$, we observe $\text{MSE} \left[ \hat{\theta} \right] = \text{var} f \hat{\theta}$.

   b. By the equation $\text{MSE} \left[ \hat{\theta} \right] = \text{Bias} \left[ \hat{\theta} \right]^2 + \text{Var} \left[ \hat{\theta} \right]$, we observe $\text{MSE} \left[ \hat{\theta} \right] > \text{var} f \hat{\theta}$.

2. **Question 8.14.**

   a. Note that the density of $\hat{\theta}$ is $\int_0^n (\hat{\theta}/\theta)^{an} d\theta = \theta \int_0^n \alpha^n \theta dz = \theta \frac{\alpha^n}{\alpha+1} \theta^{\alpha+1}$. Then

   \[
   E \left[ \hat{\theta} \right] = \int_0^\theta \alpha^n (\hat{\theta}/\theta)^{an} d\theta = \theta \int_0^\theta \alpha^n \theta dz = \theta \frac{\alpha^n}{\alpha+1} \theta^{\alpha+1} < \theta. \text{ So } \hat{\theta} \text{ is biased.}
   \]

   b. As above, $E \left[ \frac{\alpha n+1}{\alpha n} \hat{\theta} \right] = \theta$, and hence $\frac{\alpha n+1}{\alpha n} \hat{\theta}$ is unbiased.

   c. The second moment of $\hat{\theta}$ is $E \left[ \hat{\theta}^2 \right] = \int_0^\theta \alpha^n (\hat{\theta}/\theta)^{an} d\theta = \theta^2 \int_0^\theta \alpha^n \theta dz = \theta^2 \frac{\alpha^n}{\alpha+2} \theta^{\alpha+2}$, and so $\text{Var} \left[ \hat{\theta} \right] = \theta^2 \left( \frac{\alpha^n}{\alpha+2} - \frac{\alpha^n}{\alpha+1} \right)^2$, and

   \[
   \text{MSE} \left[ \hat{\theta} \right] = \theta^2 \left( \frac{\alpha^n}{\alpha+2} - \frac{\alpha^n}{\alpha+1} \right)^2 + \left( \frac{\alpha^n}{\alpha+1} - 1 \right)^2 = \frac{2}{\alpha^n + \alpha n + 2}.
   \]

3. **Question 8.22.**

   The point estimate is the sample average 7.2%. A 95% confidence interval for this average is $7.2\% \pm 5.6\% \times 2/\sqrt{199} = 7.2\% \pm 0.79\% = (6.41\%, 8.00\%)$. The 2 comes from the argument in the text on p. 400, using Tchebyshoff’s inequality to bound the center .75 of the distribution; you might also use 1.96 here, from the normal table or the $t$ table with 199 degrees of freedom.

4. **Question 8.34.**

   Conventionally, you would estimate $\lambda$ by $\hat{\lambda} = \bar{Y}$, but, given the relation between mean and variance, you could also estimate $\lambda$ by $n$ times an estimate of the variance of the $Y_i$. Hence one could estimate $\lambda$ by $\hat{\lambda} = n \sum_{i=1}^n (Y_i - \bar{Y})^2/(n-1)$. Similarly, a standard error for $\bar{Y}$ would most conventionally be $(\sqrt{\sum_{i=1}^n (Y_i - \bar{Y})^2/(n-1)})/\sqrt{n}$, or $\sqrt{\bar{Y}/n}$.

5. **Question 8.43.**

   a. $P \left[ Y_{(n)} \leq y \right] = P \left[ Y_j \leq y \forall j \right] = \prod_{j=1}^n P \left[ Y_j \leq y \right] = (y/\theta)^n$. So

   \[ P \left[ U \leq u \right] = P \left[ Y_{(n)} \leq u \theta \right] = u^n. \]

   b. You are asked to generate a lower confidence bound. Recall from the discussion in class that when the pivotal quantity decreases in the parameter, the lower confidence bound is generated by the upper quantile of the pivotal quantity. That is,

   \[ .95 = P \left[ U \leq .95^{1/n} \right] = P \left[ Y_{(n)}/\theta \leq .95^{1/n} \right] = P \left[ \theta/Y_{(n)} \geq .95^{-1/n} \right] = P \left[ \theta \geq Y_{(n)}.95^{-1/n} \right], \]

   and $(Y_{(n)}.95^{-1/n}, \infty)$ is a confidence interval for $\theta$. 

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Homework 1 Solutions,
6. Question 8.72.

a. The variance of the estimator of the difference is \( \sqrt{\pi_m(1-\pi_m)/1000 + \pi_f(1-\pi_f)/1000} \).

Worst case for standard error is \( \sqrt{.25/1000 + .25/1000} = 0.0223 \). The bound on the error is the half-width of the 95% confidence interval. This is \( z_{.05}/2 \cdot 0.0223 = 1.96 \times 0.0223 = 0.0438 \).

b. Since the problem doesn’t specify the way the interviewees will be divided between men and women, assume that they are evenly split; call the common value \( n \). Then we want \( 1.64\sqrt{.25/n + .25/n} = 0.02 \). Hence \( 1.16/\sqrt{n} = 0.02 \), or \( n = 1.16^2/0.02^2 = 3364 \); interview 3364 subjects in each group.