Homework 2 Solutions,

1. Question 9.5.

Let \( Y^* = (Y_1 + Y_2)/2 \). Then \( \hat{\sigma}_2 = |(Y_2 - Y^*)^2 + (Y_1 - Y^*)^2 - 2(Y_1 - Y^*)(Y_2 - Y^*)|/2 \).

Since \( Y_1 - Y^* = -(Y_2 - Y^*) \), \( \hat{\sigma}_2 = (Y_2 - Y^*)^2 + (Y_1 - Y^*)^2 \). Hence \( \hat{\sigma}_2 \) is the usual sample variance evaluated using only the first two observations. Use theorem 7.3 to give the distribution of \( \sigma_2^2/(n-1)/\sigma^2 \) as chi-square with \( n-1 \) degrees of freedom. The table at the back of the book gives the variance of this distribution as \( 2(n-1) \), and so the variance of \( \hat{\sigma}_1 = \sigma^2 \) is \( 2/(n-1)\sigma^4 \).

Similarly, the variance of \( \hat{\sigma}_2 \) is \( 2\sigma^4 \). The efficiency of \( \hat{\sigma}_1 \) relative to \( \hat{\sigma}_2 \) is \( n-1 \).

2. Question 9.16.

Because \( \hat{\sigma}_2 \) depends only on the first two variables \( Y_1 \) and \( Y_2 \), and because the \( Y \) values are independent, the distribution of \( \hat{\sigma}_2 \) does not change as \( n \) changes. Hence the variability of \( \hat{\sigma}_2 \) about \( \sigma^2 \) does not decrease as \( n \to \infty \), and \( \hat{\sigma}_2 \) is not consistent.

3. Question 9.44.

The joint density is given by \( \prod_{i=1}^{n} y_i^{-(\alpha+1)} \alpha \beta^\alpha I(y_i \geq \beta) \). The condition that \( y_i \) exceed \( \beta \) may be ignored, since \( \beta \) is known. The joint density may then be rewritten \( \alpha^n \beta^{n\alpha} (\prod_{i=1}^{n} y_i)^{-(\alpha + 1)} I(\min y_i > \beta) = g(\alpha, \prod_{i=1}^{n} y_i) h(y_1, \ldots, y_n) \), for \( g(\alpha, t) = \alpha^n \beta^{n\alpha} t^{-(\alpha + 1)} \) and \( h(y_1, \ldots, y_n) = I(\min y_i > \beta) \). By the factorization theorem, \( \prod_{i=1}^{n} y_i \) is sufficient for \( \alpha \), with \( \beta \) known.

4. Question 9.49.

The joint density is given by \( \prod_{i=1}^{n} I(y_i < \theta)/\theta = \theta^{-n} I(\max y_i < \theta) = g(\max y_i, \theta) h(y_1, \ldots, y_n) \), for \( g(t, \theta) = I(t < \theta) \theta^{-n} \), and \( h(y_1, \ldots, y_n) = 1 \). By the factorization theorem, \( \max Y_i \) is sufficient.

5. Question 9.52.

The joint density is given by \( \prod_{i=1}^{n} 3y_i^2 \theta^{-3} I(y_i \leq \theta) = 3^n (\prod_{i=1}^{n} y_i)^2 \theta^{-3n} I(y(n) \leq \theta) g(t, \theta) h(y_1, \ldots, y_n) \) for \( g(t, \theta) = I(y(n) \leq \theta) \theta^{-3n} \) and \( h(y_1, \ldots, y_n) = 3^n (\prod_{i=1}^{n} y_i)^2 \). By the factorization theorem, \( y(n) \) is sufficient for \( \theta \).


1. The CDF of \( Y_i \) is \( \int_{0}^{y} \int_{0}^{3z^2 \theta^{-3}} dz = y^3 \theta^{-3} \). Then \( P[Y(n) \leq y] = P[Y_1 \leq y \forall i] = \prod_{i=1}^{n} P[Y_i \leq y] = y^{3n} \theta^{-3n} \), and the density is the derivative \( 3ny_{n-1}^3 \theta^{-3n} \), for \( y \in [0, \theta] \).

2. The expectation of the maximum is given by \( \int_{0}^{\theta} y y_{n-1}^3 \theta^{-3n} dy = \frac{3\theta^2}{3n+1} \). Then \( \hat{\theta} = \frac{3n+1}{3n} Y(n) \) is unbiased, and since it is a function of the sufficient statistic, will not be improved via Rao-Blackwell.