Note: The problems are taken from the Exercises in Gelman et al. (2014), 3rd edition, unless otherwise noted. For each problem, please explain your reasoning clearly. It is not acceptable to only provide your final result.

**Homework 1** (Due Thu, Feb 13):
1.1, 2.1, 2.2, 2.7, 2.16(a), 3.1, 3.10

For 2.16, also compute the posterior predictive distribution of a new observation $\tilde{y}$.

**Additional Problems:**

1. Suppose $y | \theta \sim \text{Binomial } (n, \theta)$ and $\theta \sim \text{Unif } (0, 1)$.
   
   (i) For a new observation $\tilde{y}$, calculate the posterior predictive density $p(\tilde{y} | y)$ analytically.
   
   (ii) For $n = 10$, plot the posterior predictive density $p(\tilde{y} | y)$ as $y$ varies from 0 to 10.

2. Suppose $y | \theta \sim \text{N(} \theta, \sigma^2 \text{)}$ and $\theta \sim \text{N(} \mu_0, \tau_0^2 \text{)}$. Calculate the marginal density (i.e., the prior predictive density) $p(y)$.

3. Suppose $(y_1, \ldots, y_n | \sigma^2) \sim \text{N(} \mu, \sigma^2 \text{)}$ with $\mu$ known and the prior $p(\sigma^2) \propto \sigma^{-2}$.
   
   (i) Compute the posterior of $\sigma^2$.
   
   (ii) Can you construct a posterior interval of $\sigma^2$ which matches the (frequentist) confidence interval given a confidence level (say 95%)?

4. Suppose $(y_1, \ldots, y_n | \mu) \sim \text{N(} \mu, \sigma^2 \text{)}$ with $\sigma^2$ known and the prior $p(\mu) \propto 1$. Find the posterior predictive density of a new observation $\tilde{y}$ such that $\tilde{y} | (y_1, \ldots, y_n, \mu) \sim \text{N(} \mu, \sigma^2 \text{)}$.

**Homework 2** (Due Thu, March 5, extended to Thu, April 2):
3.2, 3.3, 3.5, 5.3, 5.9, 5.13

**Homework 3** (Due Thu, April 16):

1. Consider a bivariate probability density function $\pi(x_1, x_2)$ and an unnormalized density function $q(x_1, x_2)$ as follows:
   
   $$\pi(x_1, x_2) \propto q(x_1, x_2) = \exp\{-\frac{1}{2}(x_1^2 x_2^2 + x_1^2 + x_2^2 - 2x_1 - 2x_2)\}, \quad (x_1, x_2) \in \mathbb{R}^2.$$

   (i) Derive the conditional distributions $\pi(x_1 | x_2)$ and $\pi(x_2 | x_1)$.
   
   (ii) Use Gibbs sampling to simulation from $\pi(x_1, x_2)$.

2. Consider Exercise 3.5(b).
(i) Use (random walk) Metropolis sampling to simulate from the posterior distribution. Justify your choice of the proposal distribution.

(ii) Use independence Metropolis-Hastings sampling to simulate from the posterior distribution. Use the posterior from 3.5(a) as the proposal distribution.

3. For 5.13(b), use (random walk) Metropolis sampling to simulate from the posterior distribution. Justify your choice of the proposal distribution.

Homework 4 (Due 5:00PM Eastern, Thursday, May 14):
13.5, 16.2

Additional Problems:

1. Consider logistic regression,

\[ P(y_i = 1|\beta) = \frac{e^{x_i^T\beta}}{1 + e^{x_i^T\beta}}, \quad i = 1, \ldots, n. \]

Suppose that the prior on \( \beta \) is \( N(\alpha, A) \) with some vector \( \alpha \) and matrix \( A \). Derive the normal approximation used in the R code HaasLogitMH.txt.

2. Consider linear regression with known error variance \( \sigma^2 = 1 \),

\[ y_i = x_i^T\beta + \epsilon_i, \quad \epsilon_i \sim N(0, 1), \quad i = 1, \ldots, n. \]

Denote \( Y = (y_1, \ldots, y_n)^T \) and \( X = (x_1, \ldots, x_n)^T \) (the data matrix). Suppose that the prior on \( \beta \) is \( N(\alpha, A) \) with some vector \( \alpha \) and matrix \( A \). Show that the posterior of \( \beta \) is \( N(\tilde{\beta}, \tilde{A}) \), where

\[ \tilde{\beta} = \tilde{A}(A^{-1}\alpha + X^TY), \quad \tilde{A} = (A^{-1} + X^TX)^{-1}. \]

Note: this is related to Gibbs sampling used for Bayesian probit regression with data augmentation.

3. Consider the 8-school example (Section 5.5).

(i) Draw a contour plot of the marginal posterior density \( p(\mu, \log \tau|y) \) (cf. Figure 5.3);

(ii) Derive the EM algorithm to find a mode of \( p(\mu, \log \tau|y) \);

(iii) Find the hessian at the mode of \( p(\mu, \log \tau|y) \) found in (ii).

4. Exercise 11.3: Consider only the hierarchical normal model in Section 11.6.

(i) Compute the MLEs of \((\mu, \sigma^2, \tau^2)\) using the EM algorithm. Provide your analytical formulas and R codes.
(ii) Compute the predicted values of $\theta_1, \ldots, \theta_6$ based on MLEs in (i). Provide your analytical formulas and R codes.

(iii) Present posterior histograms of $(\mu, \sigma^2, \tau^2)$ and report posterior quantiles (5%, 25%, 50%, 75%, 95%) using Gibbs sampling. Use the same prior as in Section 11.6 of the book. Provide your analytical formulas and R codes.

(iv) Present posterior histograms of $(\theta_1, \ldots, \theta_6)$ and report posterior quantiles (5%, 25%, 50%, 75%, 95%) of $(\theta_1, \ldots, \theta_6)$ using Gibbs sampling. Provide your analytical formulas and R codes.

(v) Repeat (iii) and (iv), but using Metropolis sampling in the approach discussed in Section 13.6.