Solutions to HW 4, even-numbered and extra problems

4.6

b.

$$f(y) = F'(y) = \begin{cases} 2y\mathrm{e}^{-y^2}, & y > 0 \\ 0, & y \leq 0 \end{cases}$$

c. $P(Y \geq 2) = 1 - F(2) = 1 - (1 - \mathrm{e}^{-4}) = \mathrm{e}^{-4}$

d. $P(1 < Y \leq 2) = F(2) - F(1) = (1 - \mathrm{e}^{-4}) - (1 - \mathrm{e}^{-}) = \mathrm{e}^{-1} - \mathrm{e}^{-4}$. Therefore,

$$P(Y > 1 | Y \leq 2) = \frac{P(1 < Y \leq 2)}{P(Y \leq 2)} = \frac{\mathrm{e}^{-1} - \mathrm{e}^{-4}}{1 - \mathrm{e}^{-4}}$$

4.12

a. Calculate

$$\begin{aligned} \int_{-\infty}^{\infty} f(y)\,\mathrm{d}y &= \int_{-1}^{0} .2\,\mathrm{d}y + \int_{0}^{1} (.2 + cy)\,\mathrm{d}y \\ &= .2y]_{-1}^{0} + \left[.2y + \frac{cy^2}{2} \right]_0^1 \\ &= .2 + .2 + \frac{c}{2}. \end{aligned}$$

Setting $\int_{-\infty}^{\infty} f(y)\,\mathrm{d}y = 1$, we obtain $c = 1.2$.

b.

$$\begin{aligned} F(y) &= \int_{-\infty}^{y} f(t)\,\mathrm{d}t \\ &= \begin{cases} 0, & y < -1, \\ \int_{-1}^{y} .2\,\mathrm{d}t = .2y + .2, & -1 \leq y \leq 0, \\ \int_{-1}^{0} .2\,\mathrm{d}t + \int_0^y (.2 + 1.2t)\,\mathrm{d}t = .2 + .2y + .6y^2, & 0 \leq y \leq 1, \\ 1, & y > 1. \end{cases} \end{aligned}$$

c. Omitted.

d.

$$\begin{aligned} F(-1) &= .2(-1 + 1) = 0, \\ F(0) &= .2(0 + 1) = .2, \\ F(1) &= .2(1 + 1 + 3) = 1. \end{aligned}$$

e. $P(0 \leq Y \leq .5) = F(.5) - F(0) = .2[1 + .5 + 3(.5)^2] - .2 = .25.$

f. $P(Y > .5 | Y > .1) = \frac{P(Y>.5)}{P(Y>.1)} = \frac{1-F(.5)}{1-F(.1)} = \frac{1-.45}{1-.2[1+.1+3(.1)^2]} = \frac{.55}{.774} = .71.$

g.

$$
\begin{aligned}
F_Z(z) &= P(Z \leq z) \\
&= \begin{cases} 0, & z < 0, \\ P(Y \leq .1) = .2[1 + .1 + 3(.1)^2] = .226, & 0 \leq z \leq .1, \\ P(Y \leq z) = .2 + .2z + .6z^2, & .1 \leq z \leq 1, \\ 1, & z > 1. \end{cases}
\end{aligned}
$$

h. Consider $Z = g(Y)$ as a function of Y, where

$$
g(y) = \begin{cases} 0 & \text{if } y \leq .1, \\ y & \text{if } y > .1. \end{cases}
$$

Then

$$
\begin{aligned}
E(Z) = E[g(Y)] &= \int_{-\infty}^{\infty} g(y) f(y)\,\mathrm{d}y \\
&= \int_{-\infty}^{.1} 0 \cdot f(y)\,\mathrm{d}y + \int_{.1}^{\infty} y f(y)\,\mathrm{d}y \\
&= 0 + \int_{.1}^{1} y(.2 + 1.2y)\,\mathrm{d}y \\
&= \left[.2\frac{y^2}{2} + 1.2\frac{y^3}{3} \right]_{.1}^{1} = .5 - .0014 = .4986,
\end{aligned}
$$

$$
\begin{aligned}
E(Z^2) = E[g^2(Y)] &= \int_{-\infty}^{\infty} g^2(y) f(y)\,\mathrm{d}y \\
&= \int_{-\infty}^{.1} 0 \cdot f(y)\,\mathrm{d}y + \int_{.1}^{\infty} y^2 f(y)\,\mathrm{d}y \\
&= 0 + \int_{.1}^{1} y^2(.2 + 1.2y)\,\mathrm{d}y \\
&= \left[.2\frac{y^3}{3} + 1.2\frac{y^4}{4} \right]_{.1}^{1} = .5 - .00014 = .49986.
\end{aligned}
$$

Therefore, $\text{var}(Z) = E(Z^2) - E^2(Z) = .49983 - (.4986)^2 = .2513.$

4.26

a. $E(Y) = \frac{3}{64} \int_0^4 y^3(4 - y)\,\mathrm{d}y = \frac{3}{64} \left[y^4 - \frac{y^5}{5} \right]_0^4 = 2.4,$

$E(Y^2) = \frac{3}{64}\int_0^4 y^4(4-y)\,\mathrm{d}y = \frac{3}{64}\left[\frac{4}{5}y^5 - \frac{y^6}{6}\right]_0^4 = 6.4,$

$\mathrm{var}(Y) = E(Y^2) - E^2(Y) = 6.4 - (2.4)^2 = .64.$

b. The weekly cost for CPU time is $X = 200Y$.

$E(X) = E(200Y) = 200E(Y) = 200(2.4) = 480,$

$\mathrm{var}(X) = \mathrm{var}(200Y) = 200^2\mathrm{var}(Y) = 40000(.64) = 25600.$

c. $P(X > 600) = P(Y > 3) = \frac{3}{64}\int_3^4 y^2(4-y)\,\mathrm{d}y = \frac{3}{64}\left[\frac{4}{3}y^3 - \frac{y^4}{4}\right]_3^4 = .26.$

4.38 Let Y be the time that the call comes in. Then Y is a uniformly distributed RV on the interval 0 (midnight) to 5am. The PDF $f(y) = 1/5$ for $0 < y < 5$ and 0 otherwise. Therefore,

$$P(0 < Y < 1) + P(3 < Y < 4) = \int_0^1 \frac{1}{5}\,\mathrm{d}y + \int_3^4 \frac{1}{5}\,\mathrm{d}y = .4.$$

4.40 $E(Y) = \frac{50+70}{2} = 60,$

$\mathrm{var}(Y) = \frac{(70-50)^2}{12} = \frac{100}{3}.$

4.60 Let Y be the variable "exam score."

a. $P(Y > 72) = P(Z > \frac{72-78}{6}) = P(Z > -1) = .8413.$

b. We seek c such that $.1 = P(Y > c) = P(Z > \frac{c-78}{6})$. From Table 4, $\frac{c-78}{6} = .58$, and $c = 85.68$.

c. We seek c such that $.281 = P(Y > c) = P(Z > \frac{c-78}{6})$. From Table 4, $\frac{c-78}{6} = .25$, and $c = 81.48$.

d. We seek c such that $.25 = P(Y < c) = P(Z < \frac{c-78}{6})$. From Table 4, $\frac{c-78}{6} = -.67$, and $c = 73.98$.

e. $P(Y > 84|Y > 72) = \frac{P(Y>84)}{P(Y>72)} = \frac{P(Z>\frac{84-78}{6})}{P(Z>\frac{72-78}{6})} = \frac{P(Z>1)}{P(Z>-1)} = \frac{.1587}{.8413} = .1886.$

4.70 Let X = number of earthquakes to exceed 5.0 on the Richter scale, Y = magnitude of the earthquake. Then Y is exponential with $\beta = 2.4$ and X is Binomial $(10, p = P(Y > 5))$, where

$$p = \int_5^\infty \frac{1}{2.4}\mathrm{e}^{-y/2.4}\,\mathrm{d}y = \mathrm{e}^{-y/2.4}\Big|_5^\infty = .1245$$

Therefore

$$P(X \geq 1) = 1 - P(X = 0) = 1 - (.8755)^{10} = .7354$$

4.72 Note that Y is exponential with mean 10. We calculate

$$E(Y^2) = \text{var}(Y) + E^2(Y) = 10^2 + 10^2 = 200,$$
$$E(Y^3) = \int_0^\infty \frac{y^3}{10} \mathrm{e}^{-y/10}\,\mathrm{d}y \overset{z=y/10}{=} 10^3 \int_0^\infty z^3 \mathrm{e}^{-z}\,\mathrm{d}z = 10^3\Gamma(4) = 10^3(3!) = 6000,$$
$$E(Y^4) = \int_0^\infty \frac{y^4}{10} \mathrm{e}^{-y/10}\,\mathrm{d}y \overset{z=y/10}{=} 10^4 \int_0^\infty z^4 \mathrm{e}^{-z}\,\mathrm{d}z = 10^4\Gamma(5) = 10^5(4!) = 240000.$$

Therefore,

$$\begin{aligned} E(C) &= 100 + 40E(Y) + 3E(Y^2) \\ &= 100 + 40(10) + 3(200) = 1100, \\ E(C^2) &= E[(100 + 40Y + 3Y^2)^2] \\ &= E(10000 + 8000Y + 2200Y^2 + 240Y^3 + 9Y^4) \\ &= 10000 + 8000E(Y) + 2200E(Y^2) + 240E(Y^3) + 9E(Y^4) \\ &= 10000 + 8000(10) + 2200(200) + 240(6000) + 9(240000) \\ &= 4130000. \end{aligned}$$

4.82 Let Y be the number of the 3 components operating more than 200 hours. Then because the components operate independently, Y has a binomial distribution with $n = 3$ and

$$p = \int_{200}^\infty \frac{1}{100} \mathrm{e}^{-y/100}\,\mathrm{d}y = \left[-\mathrm{e}^{-y/100}\right]_{200}^\infty = \mathrm{e}^{-2}.$$

Then

$$\begin{aligned} P(\text{ equipment operates at least 200 hours }) = P(Y \geq 2) &= \binom{3}{2} p^2(1-p) + \binom{3}{3} p^3 \\ &= 3(\mathrm{e}^{-2})^2(1-\mathrm{e}^{-2}) + (\mathrm{e}^{-2})^3 = .05. \end{aligned}$$

4.82 - extra question Since Y has a binomial distribution with $n = 3$ and $p = \mathrm{e}^{-2}$,

$$\begin{aligned} E(Y) &= np = 3\mathrm{e}^{-2} = .406 \\ \text{var}(Y) &= np(1-p) = 3\mathrm{e}^{-2}(1-\mathrm{e}^{-2}) = .351 \end{aligned}$$

4.98 We calculate

$$\begin{aligned} E(Y^2) &= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 y^2 \cdot y^{\alpha-1}(1-y)^{\beta-1}\,\mathrm{d}y \\ &= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 \cdot y^{\alpha+1}(1-y)^{\beta-1}\,\mathrm{d}y \\ &= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \left[\frac{\Gamma(\alpha+2+\beta)}{\Gamma(\alpha+2)\Gamma(\beta)}\right]^{-1} \\ &= \frac{\alpha(\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)}. \end{aligned}$$

Therefore

$$\begin{aligned} \mathrm{var}(Y) &= E(Y^2) - E^2(Y) \\ &= \frac{\alpha(\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)} - \frac{\alpha^2}{(\alpha+\beta)^2} \\ &= \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}. \end{aligned}$$

4.99 - extra question We seek $0 < c < 1$ such that

$$\begin{aligned} 10\% &= \int_c^1 \frac{\Gamma(1+2)}{\Gamma(1)\Gamma(2)} y^{1-1}(1-y)^{2-1}\,\mathrm{d}y \\ &= \int_c^1 2(1-y)\,\mathrm{d}y \\ &= \left[2y - y^2\right]_c^1 = 1 - (2c - c^2) \\ &= (1-c)^2. \end{aligned}$$

Therefore $c = 1 - \sqrt{.1} = .68$.