Problem 1.

1)

$$\begin{aligned} P(\text{ at least 1 black ball }) &= 1 - P(\text{ both white balls }) \\ &= 1 - \frac{6}{10}\frac{8}{10} \\ &= 1 - .48 \\ &= .52 \end{aligned}$$

2)

$$\begin{aligned} P(\text{ exactly 1 black ball }) = & P(\text{ black ball from box I \& white ball from box II }) + \\ & P(\text{ white ball from box I \& black ball from box II }) \\ = & \frac{4}{10}\frac{8}{10} + \frac{6}{10}\frac{2}{10} \\ = & .32 + .12 \\ = & .44 \end{aligned}$$

3)

$$\begin{aligned} P(\text{ no black ball }) &= P(\text{ both white balls }) \\ &= .48 \end{aligned}$$

4)

$$\begin{aligned} P(\text{ both black balls }) &= \frac{4}{10}\frac{2}{10} \\ &= .08 \end{aligned}$$

5)

$$\begin{aligned} & P(\text{ exactly 1 black ball } \mid \text{ at least 1 black ball }) \\ & = \frac{P(\text{ exactly one black ball })}{P(\text{ at least one black ball })} \\ & = \frac{.44}{.52} = \frac{11}{13} \end{aligned}$$

6)

$$
\begin{aligned}
&P(\text{ ball from box I is black} \mid \text{at least 1 black ball })\\
&= \frac{P(\text{ ball from box I is black })}{P(\text{ at least one black ball })}\\
&= \frac{.4}{.52} = \frac{10}{13}
\end{aligned}
$$

7)

$$
\begin{aligned}
&P(\text{ ball from box I is black} \mid \text{exactly 1 black ball })\\
&= \frac{P(\text{ ball from box I is black \& ball from box II is white})}{P(\text{ exactly one black ball })}\\
&= \frac{(.4)(.8)}{.44} = \frac{8}{11}
\end{aligned}
$$

8)

$$
\begin{aligned}
f_X(0) &= P(X=0) = P(\text{ both black ball }) = .48\\
f_X(1) &= P(X=1) = P(\text{ exactly 1 black ball }) = .44\\
f_X(2) &= P(X=2) = P(\text{ both black balls }) = .08
\end{aligned}
$$

9) Note that $Y = 3X + 1(2 - X) = 2 + 2X$. Therefore,

$$
\begin{aligned}
f_Y(2) &= P(Y=2) = P(X=0) = .48\\
f_Y(4) &= P(Y=4) = P(X=1) = .44\\
f_Y(6) &= P(Y=6) = P(X=2) = .08
\end{aligned}
$$

10) Note that

$$
\begin{aligned}
E(X) &= 0(.48) + 1(.44) + 2(.08) = .6\\
E(X^2) &= 0(.48) + 1(.44) + 4(.08) = .76\\
\text{var}(X) &= E(X^2) - E^2(X) = .76 - (.6)^2 = .4
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
E(Y) &= 2 + 2E(X) = 2 + 2(.6) = 3.2\\
\text{var}(Y) &= 4\text{var}(X) = 4(.4) = 1.6
\end{aligned}
$$

11) Z is geometric with $p = P($ at least 1 black ball $) = .52$. Therefore,

$$f_Z(z) = (.52)(.48)^{z-1}, \quad z = 1, 2, ...$$

12)

$$\begin{aligned} E(Z) &= \frac{1}{p} = \frac{1}{.52} = \frac{25}{13} = 1.92 \\ \text{var}(Z) &= \frac{1-p}{p^2} = \frac{1-.52}{(.52)^2} = 1.78 \end{aligned}$$

Problem 2.

1) Let $A = \{$ 2 black balls $\}$ and $B = \{$ box I is selected $\}$. Then

$$\begin{aligned} P(A) &= P(A|B)P(B) + P(A|\overline{B})P(\overline{B}) \\ &= \frac{\binom{4}{2}}{\binom{10}{2}}\frac{1}{2} + \frac{\binom{2}{2}}{\binom{10}{2}}\frac{1}{2} \\ &= \frac{2}{15}\frac{1}{2} + \frac{1}{45}\frac{1}{2} = \frac{7}{90} \end{aligned}$$

2)

$$\begin{aligned} P(B|A) &= \frac{P(A|B)P(B)}{P(A)} \\ &= \frac{2/30}{7/90} = \frac{6}{7} \end{aligned}$$

3)

$$\binom{20}{4}\left(\frac{7}{90}\right)^4\left(1-\frac{7}{90}\right)^{16} = .0485$$

4)

$$\begin{aligned} &\binom{20}{4}\left(\frac{7}{90}\right)^4\left(1-\frac{7}{90}\right)^{16} \\ \approx &\frac{\mathrm{e}^{-1.56}1.56^4}{4!} = .0519 \end{aligned}$$

where $\lambda = np = 20(7/90) = 1.56$.

Problem 3.

1) See the answer to 2).

2) For $x = 2, 3, 4, 5$,

$$P(X = x) = P\{\text{ exactly 1 black ball in the first } (x-1) \text{ draws }\}\times$$
$$P\{\text{ the } x\text{th draw is black ball} \mid \text{exactly 1 black ball in the first } (x-1) \text{ draws }\}$$

By Theorem of black and white balls in the case of without replacement,

$$P\{\text{ exactly 1 black ball in the first } (x-1) \text{ draws }\}$$
$$= \frac{\binom{2}{1}\binom{3}{x-2}}{\binom{5}{x-1}}$$

Given exactly 1 black ball in the first $(x-1)$ draws, there are 1 black ball in the remaining $(6-x)$ balls. Then

$$P\{\text{ the } x\text{th draw is black ball} \mid \text{exactly 1 black ball in the first } (x-1) \text{ draws }\}$$
$$= \frac{1}{6-x}$$

Therefore,

$$P(X = x) = \frac{\binom{2}{1}\binom{3}{x-2}}{\binom{5}{x-1}} \times \frac{1}{6-x}, \qquad x = 2, 3, 4, 5$$

That is,

$$P(X = 2) = \frac{\binom{2}{1}\binom{3}{0}}{\binom{5}{1}} \times \frac{1}{4} = \frac{1}{10}$$
$$P(X = 3) = \frac{\binom{2}{1}\binom{3}{1}}{\binom{5}{2}} \times \frac{1}{3} = \frac{1}{5}$$
$$P(X = 4) = \frac{\binom{2}{1}\binom{3}{2}}{\binom{5}{3}} \times \frac{1}{2} = \frac{3}{10}$$
$$P(X = 5) = \frac{\binom{2}{1}\binom{3}{3}}{\binom{5}{4}} \times \frac{1}{1} = \frac{2}{5}$$

3)

$$E(X) = 2\frac{1}{10} + 3\frac{1}{5} + 4\frac{3}{10} + 5\frac{2}{5} = 4$$
$$E(X^2) = 2^2\frac{1}{10} + 3^2\frac{1}{5} + 4^2\frac{3}{10} + 5^2\frac{2}{5} = 17$$
$$\text{var}(X) = E(X^2) - E^2(X) = 17 - 3^2 = 1$$

4)