Problem 1.

(i) A deck of 52 cards is shuffled thoroughly. What is the probability that the four aces are all next to each other?

(ii) A poker player is dealt three spades and two hearts. He discards the two hearts and draws two more cards. What is the probability that he draws two more spades?

(iii) Prove the following identity:

\[ \sum_{k=0}^{n} \binom{n}{k} \binom{m-n}{n-k} = \binom{m}{n} \]

(iv) Prove the following two identities both algebraically and by interpreting their meaning combinatorially.

a. \( \binom{n}{r} = \binom{n}{n-r} \)

b. \( \binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r} \)

Solution (i)
Let \( E = \{4 \text{ aces are next to each other}\} \).
The possible outcomes are all permutations of 52 cards, so there are 52! possible outcomes. We are interested in the event the 4 aces are next to each other somewhere in the deck. Treat the 4 aces \( \{A_h, A_c, A_s, A_d\} \equiv A \) as a single element of the deck. Then we can assume a deck of 49 cards and any ordering of this deck gives us a portion of A and the order of the other 48 cards. There are 49! possible choices of this ordering. Finally, there are 4! possible orders of aces of A.

\[ P(E) = \frac{4!49!}{52!} \]

(ii)
After the initial deal, there are 10 spades left among the 47 cards remaining.
\[ P(\text{draw 2 more spades}) = \binom{10}{2} \binom{47}{2}. \]

(iii)

\[ \sum_{k=0}^{n} \binom{n}{k} \binom{m-n}{n-k} = \binom{m}{n}; m \geq n. \]

The right hand side (RHS) is the number of subsets of size \( n \) in a set size \( m \geq n \). Suppose we have a set of \( m \) elements, then we can choose \( n \) of these elements as following:

1. Partition the set \([m]\) into a subset of size \( n \) and \( m - n \);
2. Pick some \( k = 0, 1, \ldots, n \);
3. Arbitrary \( k \), choose a subset \( A_1 \) size \( k \) from the set of \( n \) elements and a subset \( A_2 \) size \( m - k \) from subset of \( m - n \) elements. Put \( A = A_1 \cup A_2 \).

Then number of possible choice of \( A_1 \) is \( \binom{n}{k} \), and number of possible choice of \( A_2 \) is \( \binom{m-n}{n-k} \).

The picture looks like this: let \( m = 10, n = 6 \); take \( k = 4 \),
1 2 3 4 5 6 ∥ 7 8 9 10
choose 4 from the first 6 numbers and 2 from the last 4,
2 4 5 6 ∥ 8 10
then take union \{2, 4, 5, 6, 8, 10\}.
Since \( k \) is arbitrary and the possible subsets for different \( k \) are all distinct (Why?), the number of possible subset of \( n \) inside \( m \) is

\[ \sum_{k=0}^{n} \binom{n}{k} \binom{m-n}{n-k} \]

(iv)

a) By definition, this means that choosing \( r \) elements of \([n]\) to include in a subset is equivalent to choosing the \( n - r \) elements to exclude.

b)

\[ \binom{n-1}{r-1} + \binom{n-1}{r} = \frac{(n-1)!}{(r-1)!(n-r-1)!} + \frac{(n-1)!}{(n-r)!(r-1)!} \]

\[ = \frac{(n-1)!}{(r-1)!(n-r-1)!} \left[ \frac{1}{n-r} + \frac{1}{r} \right] \]

\[ = \frac{(n-1)!}{(r-1)!(n-r-1)!} \left[ \frac{r + n - r}{r(n-r)} \right] \]

\[ = \binom{n}{r}. \]
Explanation: choose a subset of size $r$ from $[n]$ in one of two ways. Assume $n$ as the 'special' element. So the set is 

$$\{1, 2, 3, \ldots, n-1, n^*\}.$$ 

Now, we could either:

(a) include $n$ in our set, in which case we need to choose $r-1$ elements from $n-1$, \(\binom{n-1}{r-1}\); 

(b) not include $n$, and choose $r$ from the remaining $n-1$.

Problem 2. Consider rolling 3 (n+1)-sided dice, one Red, one White, and one Blue. By considering two different ways to count things, show that the number of ways that the Red die can show strictly more spots than both the White and Blue ones (as would be the case for $R=6$, $W=2$, $B=5$ but not for $R=9$, $W=2$, $B=9$) equals

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{1}{3}[(n + 1)^3 - (n + 1) - 3 \frac{n(n + 1)}{2}] = \frac{n(n + 1)(2n + 1)}{6}$$

Solution

Let $R$ denote number of spots on Red dice; $W$ for White; $B$ for Blue. We are interested in the number of outcomes in the set

$$E = \{R > W, R > B\} = \{R > W\} \cap \{R > B\}.$$ 

We can count in at least two ways:

1. Suppose $R = k$ for some $k = 1, 2, \ldots, n+1$. Then we have $(k-1)$ possible outcomes on white dice such that $R > W$; $(k-1)$ possible outcomes on blue dice such that $R > B$. By multiplication rule, there are $(k-1)^2$ possible outcomes in the event $R = k$. We get the total by summing over $k$:

$$\sum_{k=1}^{n+1} (k-1)^2 = 0^2 + 1^2 + \cdots + n^2.$$ 

2. Alternatively, we count the number of outcomes that are not in $E$ and subtract from the total number of outcomes $(n + 1)^3$. To do this, we only need consider outcomes for which one of the dice is greater than each of the other two; by symmetry, exactly one-third of these events will have the red die as the one showing the most. So there are $(n + 1)^3$ possible outcomes of which $n + 1$ have $B = W = R$, so we eliminate those outcomes. Of the remaining outcomes, we must exclude those for which two dice are equal and the other is less than the first two. These are outcomes in the set: 

3
\{B < R = W\} \cup \{R < B = W\} \cup \{W < R = B\}.

There are \(\sum_{k=1}^{n+1} (k - 1) = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}\) elements in each of \{B < R = W\}, \{R < B = W\}, \{W < R = B\}. Total are \(\frac{3n(n+1)}{2}\) outcomes.

Hence, we have

\[\#E = \frac{1}{3} \left[ (n+1)^3 - (n+1) - \frac{3n(n+1)}{2} \right].\]

**Problem 3.** The instruction "find expressions for" means "write in set-theoretic notation" (in the simplest possible way). For example, the answer to part (a) is that the event "only E occurs" is \(E \cap F^c \cap G^c\). For this question you may just write down the answers, without giving the justification. Let \(E\), \(F\), and \(G\) be three events. Find expressions for the events so that of \(E\), \(F\), and \(G\):

1. only \(E\) occurs;
2. both \(E\) and \(G\) but not \(F\) occur;
3. at least one of the events occur;
4. at least two of the events occur;
5. all three occur;
6. none of the events occur;
7. at most one of them occurs;
8. at most two of them occur;
9. exactly two of them occur;
10. at most three of them occur.

**Solution**

1. \(E \cap F^c \cap G^c\)
2. \(E \cap F^c \cap G\)
3. \(E \cup F \cup G\)
4. \((E \cap F) \cup (E \cap G) \cup (F \cap G)\)
5. $E \cap F \cap G$
6. $E^c \cap F^c \cap G^c$
7. $(E^c \cap F^c \cap G^c) \cup (E \cap F \cap G \cap G) \cup (E^c \cap F \cap G \cap G)
8. (E \cap F \cap G)^c = E^c \cup F^c \cup G^c
9. (E \cap F \cap G) \cup (E \cap F \cap G) \cup (E^c \cap F \cap G)
10. $\Omega$

Problem 4. Let $\Omega$ be a finite set, $\mathcal{E} = 2^\Omega$ and $P$ be a probability measure on $(\Omega, \mathcal{E})$.

(i) Show that $\#2^\Omega = 2^{\#\Omega}$
(ii) if $A \subset B \subset \Omega$, then $P[B - A] = P[B] - P[A]$.
(iii) Suppose $A, A_1, A_2, ...$ are events such that $\lim_{n \to \infty} P[A_n] = 0$. Show that if $A \subseteq A_n$ for every $n \geq 1$, then $P[A] = 0$.

If $A_1, A_2, ..., A_n$ are subsets of $\Omega$, show that
(iv) $P(\bigcup_{i=1}^n A_i) \leq \sum_{i=1}^n P(A_i)$;
(v) Verify the following extension of the addition rule (a) by an appropriate Venn diagram and (b) by a formal argument using the axioms of probability.

$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$

(vi) $P(\bigcup_{i=1}^n A_i) = \sum_{S \subseteq [n]} (-1)^{\#S + 1} P(\bigcap_{i \in S} A_i)$

Solution

(i) $\#2^\Omega = \#$ subsets of $\Omega$. Suppose $\#\Omega = n$, then,

\[
\#2^\Omega = \sum_{k=0}^n \sum_{A \subseteq \Omega : \#A = k} 1
\]

\[
= \sum_{k=0}^n \#\{A \subset \Omega : \#A = k\}
\]

\[
= \sum_{k=0}^n \binom{n}{k} 1^k 1^{n-k}
\]

\[
= (1 + 1)^n
\]

\[
= 2^n
\]

\[
= 2^{\#\Omega}
\]
(ii) \( A \subset B \subset \Omega \Rightarrow P(B - A) = P(B) - P(A) \).

Proof: \( B - A = B \cap A^c \) and \( B = (B \cap A^c) \cup (B \cap A) \) (disjoint union).

Moreover, \( A \subset B \Rightarrow B \cap A = A \).

We have

\[
B = A \cup (B \cap A^c)
\]

and

\[
P(B) = P(A \cup (B - A)) = P(A) + P(B - A)
\]

by PM(3).

(iii) \( A \subseteq A_n \Rightarrow P(A) \leq P[A_n] \)

Suppose \( P[A] = \epsilon \neq 0 \), since \( \lim_{n \to \infty} P[A_n] = 0 \), we can always find some \( A_k \), where \( P[A_k] = \delta < \epsilon \). This contradicts \( P[A] \leq P[A_n] \).

(iv) \( P(\bigcup_{i=1}^{n} A_i) \leq \sum_{i=1}^{n} P(A_i) \)

Proof: For each \( i = 2, \ldots, k \), put \( A_i^* = A_i - A_{i-1}^* \) and initially with \( A_1^* = A_1 \). Then \( \bigcup A_i = \bigcup A_i^* \) and \( A_i^* \subset A_i \) for all \( i \). By PM(2) and (iii) above, we have \( P(A_i^*) \leq P(A_i) \) for each \( i \). Also by a fact provided in class (that follows from PM(3)), we have

\[
P(\bigcup A_i) = P(\bigcup A_i^*) = \sum_{i=1}^{k} P(A_i^*) \leq \sum_{i=1}^{k} P(A_i).
\]

(v) Proof:

\[
\therefore P(A \cup B) = P(A) + P(B) - P(A \cap B)
\]

\[
\therefore P(A \cup B \cup C) = P((A \cup B) \cup C)
\]

\[
= P(A \cup B) + P(C) - P((A \cup B) \cap C)
\]

\[
= P(A) + P(B) - P(A \cap B) + P(C) - P((A \cap C) \cup (B \cap C))
\]

\[
= P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)
\]

(vi) Will be shown in class.