1. Homework 2: Poisson processes

Problem 1.1. For $\nu, \mu > 0$, let $N := \{N_t\}_{t \geq 0}$ and $M := \{M_t\}_{t \geq 0}$ be independent Poisson processes with intensities $\nu, \mu$, respectively. Let

$$\tau := \min\{t \geq 0 : N_t = 1\}$$

denote the time of the first occurrence in $N$.

(i) What is the distribution of $M_\tau$?

(ii) What is the distribution of $N_{2\tau} - N_\tau$?

Problem 1.2 (Record values). Let $U := (U_1, U_2, \ldots)$ be i.i.d. Uniform[0,1] random variables. We define the record (low) values of $U$ by

$$\sigma(1) := 1 \text{ and } \sigma(n + 1) := \min\{k \geq 1 : U_k < U_{\sigma(n)}\}.$$  

(i) Define $W_n := U_{\sigma(n)}/U_{\sigma(n-1)}$, the relative decrease in record low between the $n$th and $(n-1)$st record low value. Show that $W_n \sim $ Uniform[0,1], for all $n \geq 1$, and $W_n$ is independent of $(\sigma(n), U_{\sigma(n-1)})$.

(ii) Show that the distribution of $U_{\sigma(n)}$ is the same as $\prod_{i=1}^{n} U_i$, the product of $n$ independent Uniform[0,1] random variables.

Now, let $X := (X_1, X_2, \ldots)$ be i.i.d. Exponential(1) random variables and define the record high value times by

$$\nu(1) := 1 \text{ and } \nu(n) := \min\{k \geq 1 : X_k > X_{\nu(n)}\}.$$  

(iii) Show that $X_{\nu(1)}, X_{\nu(2)}, \ldots$ is a time-homogeneous rate-1 Poisson process.

Problem 1.3. Let $0 = T_0 < T_1 < T_2 < \cdots$ be the arrival times of a Poisson process $N := \{N_t\}_{t \geq 0}$ with intensity $\lambda > 0$. For any $t > 0$, let $A_t := t - T_{N(t)}$ be the time since the last arrival and $R_t := T_{N(t)+1} - t$ be the residual lifetime.

(i) Show that the law of $R_t$ is Exponential($\lambda$), for every $t > 0$.

(ii) Find the distribution of $A_t$.

(iii) Show that $A_t$ and $R_t$ are independent.

(iv) Show that $A_t$ converges in law as $t \to \infty$.

(v) Deduce that $A_t + R_t$ converges in law as $t \to \infty$, and observe that the limiting distribution is not Exponential($\lambda$). Explain the apparent paradox.

(vi) Fix $b > 0$ and let $\nu(b) := \min\{t \geq 0 : A_t = b\}$. What is the distribution of $\nu(b)$? (Alternatively, find its Laplace transform.)
Problem 1.4. Let $W_n$ be a sequence of non-negative integer-valued random variables for which

$$\lim_{n \to \infty} \mathbb{E}\left(\frac{W_n}{k}\right) = \frac{\lambda^k}{k!}, \quad k = 1, 2, \ldots,$$

for some $\lambda > 0$.

(i) Show that $W_n$ converges in law to $\text{Poisson}(\lambda)$.

For the rest of this problem, let $W_n$ denote the number of fixed points in a random permutation of $[n] := \{1, \ldots, n\}$.

(ii) Show that, for any $k \geq 1$,

$$\mathbb{E}\left(\frac{W_n}{k}\right) = \sum_{A \subseteq [n]: \#A = k} \mathbb{P}\{A \text{ are fixed points of random permutation}\}.$$

(iii) Conclude that, as $n \to \infty$, $W_n$ converges in law to the Poisson distribution with mean 1.